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# INITIAL-BOUNDARY VALUE PROBLEMS FOR HYPERBOLIC EQUATIONS AND THEIR DIFFERENCE APPROXIMATION WITH CHARACTERISTIC BOUNDARY

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This work consists of two parts. In the first, we consider initial-boundary value problems for strictly hyperbolic systems with constant coefficients; $u_t + Au_{x_1} + \Sigma B_j u_{x_j} = F$, $j = 2,3...,m$, in the quarter space $x_1, t \geq 0, \cancel{-\infty < x_j} < \infty$, in the case of characteristic boundary, i.e. $\det A = 0$. By using the technique of $\lambda$-matrix, we obtain an a priori estimate, which assures the continuous dependence of the solution on the inhomogeneous terms of the equation. This work generalizes the former results of Majda and Osher (1975) also to the non-

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nonsymmetrical case, simplifies their proofs and removes some of their assumptions. In Part II, we develop stability theory for Burstein difference scheme approximating the above problem (m=2) with additional assumption det $(A\alpha+B_2B)=0$. Particularly, the problem of constructing the Kreiss symmetrizer for general multidimensional dissipative approximations is resolved, thus removing the only obstacle in developing stability theory for such approximations in the noncharacteristic case.

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TECHNICAL REPORT

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## 0. Introduction

Consider a first order system of partial differential equations

$$Lu(x,t) = \frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x_1} + \sum_{j=2}^{m} B_j \frac{\partial u}{\partial x_j} = F(x,t)$$

with constant coefficients. Here $u(x,t) = (u^{(1)}(x,t),\dots,u^{(n)}(x,t))'$ is a vector function of the real variables $(x,t) = (x_1,\dots,x_m,t)$ and $A, B_j$ are constant square matrices of order $n$. We assume that (0.1) is strictly hyperbolic, i.e. for all real $\omega = (\omega_1,\omega_-)$, $\omega_- = (\omega_2,\dots,\omega_m)$ with $|\omega| \neq 0$, the eigenvalues of the matrix $iA\omega_1 + iB(\omega_-)$, $B(\omega_-) = \Sigma B_j\omega_j$, are imaginary and distinct. We assume that $A$ is singular and has the form

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & A_I & 0 \\ 0 & 0 & A_{II} \end{pmatrix}, \quad \text{where} \quad A_I = \begin{pmatrix} a_2 & & 0 \\ & \ddots & \\ 0 & & a_\ell \end{pmatrix} > 0, \quad A_{II} = \begin{pmatrix} a_{\ell+1} & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} < 0.$$

The vector $u(x,t)$ is then partitioned as $u = (u^{(1)}, u_I, u_{II})'$.*) We denote by $R_0$ the half space : $x_1 \geq 0$, $-\infty < x_j < +\infty$ ; $j = 2,3,\dots,m$, by $R_-$ the $m-1$ dimensional real space of the vectors $x_- = (x_2,\dots,x_m)$ and by $R^+$ the half line $t \geq 0$.

We study the mixed initial boundary value problem

$$\text{(0.2)} \qquad \begin{aligned} &(A)\ Lu = F \quad \text{in} \quad R_0 \times R^+ \\ &(B)\ u(x,0) = f(x) \quad \text{in} \quad R_0 \\ &(C)\ Su(0,x_-,t) = g(x_-,t) \quad \text{in}\ R_- \times R^+. \end{aligned}$$

The boundary operator $S$ is a constant $(\ell-1)\times n$ matrix such that

---

*) Here and henceforth we use the transposition symbol ' in the following sense: if $A,B,C,\dots$ are vectors or matrices then $(A,B,C,\dots)'$ replaces the usual $(A',B',C',\dots)'$.

$$S(\mathrm{Ker}\,A) = 0\,. \tag{0.3}$$

For a domain G define

$$\|u\|_{\eta,G} = \| e^{-\eta t/2} u \|_{L_2(G)}\,.$$

Consider the problem (0.2) with f = 0. The main objective in study of this problem is to prove that under the uniform Kreiss condition the a priori estimate

$$\eta \, \|u(x,t)\|^2_{\eta,R_0\times R^+} + \| Au(0,x_-,t) \|^2_{\eta,R_-\times R^+} \leq K\Big(\| g(x_-,t)\|^2_{\eta,R_-\times R^+} + \frac{\|F(x,t)\|^2_{\eta,R_0\times R^+}}{\eta}\Big) \tag{0.4}$$

holds for any $\eta>0$.
Throughout this paper we denote by K as well as by $\delta$ different positive constants. The above problem was completely investigated by Majda and Osher in [1]. Our work consists of two parts. In Part I we use some concepts from the theory of $\lambda$-matrices partially introduced by Gochberg and Rodman in [4] to reprove the above estimate. The methods of $\lambda$-matrices theory enable us to simplify the proof and to remove some of the assumption in [1].

In Part II the same methods are used in investigation of stability of so-called Burstein difference approximation applied to the problem (0.2). We restrict ourselves to the two space-dimensional case and additional assumption that the determinant $|A\omega_1+B\omega_2|$ is identically zero for any $\omega_1$ and $\omega_2$. This and other technical assumptions of this work are satisfied, for example, in the case of the acoustic equations. It should be noted that Gustafsson, Kreiss and Sundstrom developed in [1] a stability theory of general difference approximations for initial boundary value problems in the case of one-space dimension and non-characteristic boundary. As far as we know, there is no such theory for several dimensional case even for non-characteristic boundary. There are two main difficulties in our investigation. The first one consists of searching for the block structure of some $\lambda$-matrix depending on parameters near a point where this matrix is non-regular. Such situation occurs because the boundary is characteristic. The second one is construction of the Kreiss symmetrizer for a block which is a perturbation of a single Jordan cell. For a general differential case such a symmetrizer was built by Kreiss in [2]. But for difference approximations this problem was resolved in [3] only in the one dimensional case for strictly nondissipative schemes. However, when the several dimensional case is considered, such a problem arises also for dissipative schemes. Being concerned with specific difference approximation we are able to provide a detailed analysis of stability. But the same methods may be applied to other difference approximations, the amplification matrix of which is a polynomial of some linear combination of the matrices A and $B_j$ from (0.1).

# Part I. Differential Equation

## 1. Definitions, Assumptions, Statements of Results.

Let us apply to the problem (0.2) with f = 0 a Fourier transform in $x_-$ with dual variable $\omega_- \in R_-$ and Laplace transform in t with dual variable $s = \xi + i\eta$, $\eta > 0$ and $-\infty < \xi < \infty$. Denote the transforms of u and F by $\hat{u}$ and $\hat{F}$. Then problem (0.2) is converted to

$$\text{(A)}\quad \hat{L}\left(\frac{d}{dx_1}, \omega_-, s\right)\hat{u}(x_1) = \left(sI + A\frac{d}{dx_1} + iB(\omega_-)\right)\hat{u}(x_1) = \hat{F}(x_1)$$

(1.1)

$$\text{(B)}\quad S\hat{u}(0) = \hat{g}.$$

Estimate (0.4) is equivalent to an estimate

$$(1.2)\qquad \mathrm{Re}s\cdot\|\hat{u}(x_1)\|^2_{R^+} + |A\hat{u}(0)|^2 \leq K\left(|\hat{g}|^2 + \frac{\|\hat{F}(x_1)\|^2_{R^+}}{\mathrm{Re}s}\right)$$

for any real $\omega_-$ and complex s with Res>0.

For simplicity of notation we remove the symbol ^ from L,u and g and replace $\omega_-$ by $\omega$ and $x_1$ by x. We also omit the symbol $R^+$ in the notion of the norm. With the differential operator $L\left(\frac{d}{dx}, \omega, s\right)$ is connected a $\lambda$-matrix $L(\lambda,\omega,s) = sI + A\lambda + iB(\omega)$. Its characteristic polynomial has the form

$$(1.3)\qquad |L(\lambda,\omega,s)| = \sum_{j=0}^{n-1} a_j(\omega,s)\lambda^j \quad \text{with}$$

$$a_{n-1}(\omega,s) = (s+ib_{11}(\omega))|A_I||A_{II}|.$$

Here $b_{11}(\omega)$ is a left upper element of $B(\omega)$ and is a linear function of $\omega$. It follows from strict hyperbolicity that for real s and $\lambda$ and imaginary $\omega$ the determinant $|L(\lambda,\omega,s)|$ is real. Therefore also $b_{11}(\omega)$ is real for real $\omega$. Without loss of generality, we may assume that $b_{11}(\omega) = 0$ (one should replace the parameter s by $s = s + ib_{11}(\omega)$) For s = 0 the highest term in (1.3) vanishes. We consider two cases:

Case 1: the polynomial $|L(\lambda,\omega,0)|$ vanishes for any $\lambda$ and $\omega$.

This is designated in [[illegible]] the case of bounded eignevalues.

Case 2: the polynomial $|L(\lambda,\omega,0)|$ does not vanish identically according to $\lambda$ for any value of $\omega \in R^{n-1}$, $\omega \neq 0$.

This is the case of unbounded eigenvalues described in [.].

Let us consider initially the first case.

It follows from strict hyperbolicity that the kernel of $(A\lambda + iB(\omega))$ is one dimensional for imaginary $\lambda$. We make the first

Assumption 1.1. $\dim \mathrm{Ker}(A\lambda + iB(\omega))$ [illegible] 1 for any complex $\lambda$ and complex $\omega$, $|\lambda| + |\omega| \neq 0$.

Denote by $V_0(\omega) = \underset{\lambda\in C}{\mathrm{Sp}}\, \mathrm{Ker}(A\lambda+iB(\omega))$ - the space generated by the kernels of $A\lambda+iB(\omega)$ for fixed $\omega$ and all complex $\lambda$.

We make the second

Assumption 1.2. $\dim V_0(\omega) = \frac{n+1}{2}$ for any complex $\omega \neq 0$.

Under assumption 1.1 we prove in subsection 3.3 the following important result stated there as Theorem 3.5: if for some real $\omega \neq 0$ there is a boundary condition (1.[illegible]) (B) such that the problem is properly posed, i.e. estimate (1.[illegible]) holds, then the matrices $A$ and $B_\omega$ satisfy the assumption 1.2. Thus assumption 1.2 is necessary for the well-posedness of the problem (1.[illegible]). Since for symmetric systems such boundary condition exists, for such systems assumption 1.2 is already satisfied.

For the case $n = 3$, it may be easily verified that if $A$ and $B(\omega)$ as well as $A^*$ and $B^*(\omega)$ have no common kernel for any complex $\omega \neq 0$, then assumptions 1.1 and 1.2 are fulfilled.

It may be also proved that if $A$ and $B(\omega)$ (for real $\omega \neq 0$) are symmetric and have common kernel, then assumption 1.1 is not true (for the same $\omega$). We conjecture that the converse is also true, i.e. if $A$ and $B_\omega$ are symmetric and for any real $\omega \neq 0$ the matrices $A$ and $B(\omega)$ have no common kernel, then assumption 1.1 is true.

In the second case no additional assumption is required.

Let us return to the problem (1.[illegible]) in the general case of bounded or unbounded eigenvalues. Following [[illegible]] we define $\varphi \in L_2(R^+)$ as an eigenfunction of the problem (1.1) corresponding to the eigenvalue $s$, with $\operatorname{Re}s>0$ if $\varphi$ is solution of the homogeneous equation:

$$A\frac{d\varphi}{dx_1}+s\cdot\varphi+iB(\omega)\varphi=0 \tag{1.4}$$

with boundary condition

$$S\varphi(0)=0\ . \tag{1.5}$$

It may be shown as in [2] that for $\operatorname{Re}s>0$ the characteristic equation

$$|L(\lambda,\omega,s)|=0 \tag{1.6}$$

has precisely $\ell-1$ eigenvalues $\lambda$ with $\operatorname{Re}\lambda<0$ and $n-\ell$ ones with $\operatorname{Re}\lambda>0$ Although the matrix $A$ is singular, the determinant $|L(\lambda,\omega,s)|$ does not vanish for all $\lambda$ if $s\neq 0$. Therefore if $\operatorname{Re}s>0$, we may apply to the equation (1.4) the elementary theory of ordinary linear differential equations. Thus, equation (1.4) has exactly $\ell-1$ linearly independent solutions

$$\varphi_1(x,\omega,s),\ \varphi_2(x,\omega,s),\dots,\varphi_{\ell-1}(x,\omega,s)\quad\text{in}\quad L_2(0\leq x<\infty). \tag{1.7}$$

Let these solutions be orthonormalized at $x=0$. Denote

$$N(\omega,s)=S[\varphi_1,\varphi_2,\dots\varphi_{\ell-1}]_{x=0}\ . \tag{1.8}$$

Then the uniform Kreiss condition (UKC) is stated:

(UKC) There exists a constant $\delta>0$ such that $|N(\omega,s)|\geq\delta$ for any $(\omega,s)$ with $\operatorname{Re}s>0$.

Given a vector $\varphi=(\varphi^{(1)},\varphi^{(2)},\dots,\varphi^{(n)})'$ we define $\bar\varphi=(\varphi^{(2)},\varphi^{(3)},\dots,\varphi^{(n)})'$. From (0.3) follows that $S\varphi$ actually depends on $\bar\varphi$. Orthonormalizing the vectors $\bar\varphi_j(0,\zeta)$ instead of $\varphi_j(0,\zeta)$ we define

$$\bar N(\omega,s)=S[\bar\varphi_1,\bar\varphi_2,\dots,\bar\varphi_{\ell-1}]_{x=0}\ . \tag{1.9}$$

Then the modified uniform Kreiss condition, denoted as $\overline{\text{(UKC)}}$, is formulated:

$\underline{(\overline{\mathrm{UKC}})}$ There exists a constant $\delta>0$ such that $|\bar{N}(\omega,s)| \geq \delta$ for any $(\omega,s)$ with $\mathrm{Re}\,s>0$.

Majda and Osher in [1] used the condition $(\overline{\mathrm{UKC}})$ and called it a uniform Kreiss condition.

Let us denote $\zeta = (\omega,s)$, $\omega' = \omega/|\lambda|$, $s' = s/|\lambda|$, $\zeta' = (\omega',s')$.

Let

(1.10) $$\Omega(\zeta_0') = \{\zeta \,|\, |\zeta'-\zeta_0'|<\varepsilon \text{ and } \omega' \text{ real}\}$$

be a conical neighbourhood of the point $\zeta_0' = (\omega_0',s_0')$ with real $\omega_0'$. It will be shown in Section 3 for the case of bounded eigenvalues that for any $\zeta_0'$ with $\mathrm{Re}\,s_0' \geq 0$ including $s_0' = 0$ there is some neighbourhood $\Omega(\zeta_0')$ such that the solutions in (1.7) are defined for any $\zeta\in\Omega(\zeta_0')$ with $\mathrm{Re}\,s>0$ and the vectors $\varphi_j(0,\zeta)$ depend on $\zeta'$ only, are continuous functions of $\zeta'$ at the point $\zeta_0'$ and are independent at this point. Moreover, the shortened vectors $\bar{\varphi}_j(0,\zeta_0')$ are also independent. Therefore the determinants $|N(\zeta)|$ and $|\bar{N}(\zeta)|$ depend actually on $\zeta'$ and are continuous at the point $\zeta_0'$.

For a fixed $\omega_0$, $s_0$ with $\mathrm{Re}\,s_0>0$ is an eigenvalue of the problem (1.1) iff

(1.11) $$|\bar{N}(\zeta_0)| = 0.$$

As in [2] we define $s_0$ with $\mathrm{Re}\,s_0 = 0$ as a generalized eigenvalue iff (1.11) holds for some point $\zeta_0 = (\omega_0,s_0)$ and $|\zeta_0| \neq 0$.

In the case of bounded eigenvalues we may replace the matrix $\bar{N}(\zeta)$ by $N(\zeta)$ in the definition of the eigenvalues and generalized eigenvalues. The conditions (UKC) and $(\overline{\mathrm{UKC}})$ are therefore equivalent and may be formulated:

(1.12) The problem (1.1) has no eigenvalues or generalized eigenvalues $s$ with $\mathrm{Re}\,s\geq 0$.

Unfortunately, in the case of unbounded eigenvalues the above conditions are not equivalent. For $\zeta_0' = (\omega_0',0)$ the vectors $\varphi_j(0,\zeta')$ are, generally speaking, not continuous at the point $\zeta_0'$. However, we shall see in Section 4 that the shortened vectors $\bar{\varphi}_j(0,\zeta')$ are still continuous. Therefore it is possible to define the generalized eigenvalues by using (1.11), and $(\overline{\mathrm{UKC}})$ may be formulated as in (1.12). The main result of this work is

Theorem 1. The condition $(\overline{\text{UKC}})$ is necessary and sufficient for the estimate (1.2) to hold.

Thus we extend theorem 1 in [1] also to non-symmetric systems at least in the case of a half-space and constant coefficients. For the case of unbounded eigenvalues assumption 1.6 in [1] may be dismissed and for bounded eigenvalues assumption 1.10 in [1] about singular block structure is replaced by the natural assumption 1.1 and additional necessary assumption 1.2 for non-symmetric systems.

In [1] there is given also a counter-example (B1, p. 631) of a problem (0.2) with non-symmetric matrices A and B such that $(\overline{\text{UKC}})$ is satisfied, but estimate (0.4) is false. This is the case of bounded eigenvalues, but the condition (UKC) is not fulfilled. The reason of this seeming contradiction lies in the fact that the matrices A and B have common kernel and do not satisfy the assumption 1.2.

We summarize now the contents of this part. It consists of four sections. In Section 2 we introduce some concepts from the theory of $\lambda$-matrices and prove some lemmas which are useful also in the Part II.

In Section 3 the case of bounded eigenvalues is investigated and in the same time the above mentioned theorem 3.5 is proved.

In Section 4 we finally consider the easiest case of unbounded eigenvalues.

# 2. $\lambda$-matrices

## 2.1. Generalized eigenvectors, spectral pairs and invariant subspaces.

Let $L(\lambda)$ be a square matrix of order n with entries, which are holomorphic functions of $\lambda$ in a domain $\Omega\subset\mathbb{C}$. Such matrix is also called a $\lambda$-matrix. The point $\lambda_0\in\Omega$ is an eigenvalue of $L(\lambda)$ if $|L(\lambda_0)| = 0$. The set of all eigenvalues of $L(\lambda)$ is called the spectrum of $L(\lambda)$ and denoted by $\sigma(L)$. The matrix $L(\lambda)$ is regular if $|L(\lambda)| \not\equiv 0$. Then for any compact $D\subset\Omega$ the set $\sigma(L)\cap D$ is finite. We say that $L(\lambda)$ is singular of order one if $|L(\lambda)| \equiv 0$ and rank $L(\lambda) = n-1$ for some $\lambda\in\Omega$. The points $\lambda\in\Omega$, where the rank $L(\lambda) < n-1$, form so called discrete spectrum of $L(\lambda)$, which is denoted by $\sigma_d(L)$. It is obvious that for any $D\subset\Omega$ the set $\sigma_d(L)\cap D$ is finite.

Let $\lambda_0\in\sigma(L)$. There exists holomorphic vector function $\varphi(\lambda)$ with $\varphi(\lambda_0) \neq 0$ such that the function $L(\lambda)\varphi(\lambda)$ vanishes at the point $\lambda_0$. Following Gochberg and Rodman in [4] we call $\varphi(\lambda)$ a root function corresponding to $\lambda_0$. The order of $\lambda_0$ as a zero of $L(\lambda)\varphi(\lambda)$ is called the multiplicity of $\varphi(\lambda)$ and the vector $\varphi^{(0)} = \varphi(\lambda_0)$ - an eigenvector of $L(\lambda)$ corresponding to $\lambda_0$. If $\varphi(\lambda)$ is a root function of $L(\lambda)$ of multiplicity q corresponding to $\lambda_0$ and

$$\varphi(\lambda) = \sum_{j=0}^{\infty} \varphi^{(j)}(\lambda-\lambda_0)^j$$

then the chain of vectors $\varphi^{(0)},\varphi^{(1)},\dots,\varphi^{(q-1)}$ is a Jordan chain of $L(\lambda)$ corresponding to the eigenvalue $\lambda_0$, and the vectors $\varphi^{(1)},\dots,\varphi^{(q-1)}$ are called generalized eigenvectors corresponding to $\lambda_0$.

A root function $\varphi_0(\lambda)$ is called a singular root function of a singular $\lambda$-matrix $L(\lambda)$ if $L(\lambda)\varphi_0(\lambda) \equiv 0$. The vector $\varphi_0(\lambda)$, when not zero, is an eigenvector of $L(\lambda)$ and is called a singular eigenvector of $L(\lambda)$ corresponding to $\lambda$. If $L(\lambda)$ is singular of order one then to any $\lambda\in\Omega$ corresponds exactly one singular eigenvector. An eigenvector corresponding to a point $\lambda_0\in\sigma_d(L)$ is called regular if it is not singular for the same $\lambda_0$. Similarly, a root function $\varphi(\lambda)$ corresponding to such point $\lambda_0$ is regular if the eigenvector $\varphi(\lambda_0)$ is regular.

Let $\varphi_1(\lambda)$ be a root function of $L(\lambda)$ of multiplicity q corresponding to an eigenvalue $\lambda_0$. We denote by $X_1(\lambda_0) = (\varphi_1^{(0)}, \varphi_1^{(1)}, \ldots, \varphi_1^{(q-1)})$ a matrix formed by the column-vectors of the corresponding Jordan chain and by $J_1(\lambda_0)$ a Jordan cell of the size q with the eigenvalue $\lambda_0$.
If $\varphi_1(\lambda), \varphi_2(\lambda), \ldots, \varphi_k(\lambda)$ are some root functions corresponding to $\lambda_0$, we form a matrix

$$X(\lambda_0) = (X_1(\lambda_0),\ X_2(\lambda_0), \ldots, X_k(\lambda_0))$$

and a corresponding Jordan matrix

$$J(\lambda_0) = \mathrm{diag}(J_1(\lambda_0), J_2(\lambda_0), \ldots, J_k(\lambda_0)),$$

where $\mathrm{diag}(J_1, J_2, \ldots, J_k)$ denotes the square block diagonal matrix whose main diagonal is given by $J_1, J_2, \ldots, J_k$. The sequence formed by the columns of $X(\lambda_0)$ will be called a Jordan sequence corresponding to $\lambda_0$, and the pair $(X(\lambda_0), J(\lambda_0))$ is a spectral pair corresponding to $\lambda_0$. In this work we often identify a matrix X with the sequence or even the set of its column-vectors. Therefore we shall call also the matrix $X(\lambda_0)$ a Jordan sequence. The space spanned by the eigenvectors $\varphi_1^{(0)}(\lambda_0), \varphi_2^{(0)}(\lambda_0), \ldots, \varphi_k^{(0)}(\lambda_0)$ is called the eigenspace of the Jordan sequence $X(\lambda_0)$ and any vector belonging to this space is called an eigenvector of $X(\lambda_0)$. If the dimension of the above eigenspace is k and any eigenvector of $X(\lambda_0)$ is not singular, then the sequence $X(\lambda_0)$ is called regular. For a regular matrix $L(\lambda)$ we replace the singular eigenvector in the above definition by 0. If $\lambda_1, \lambda_2, \ldots, \lambda_t$ are different eigenvalues of $L(\lambda)$ and $(X(\lambda_j), J(\lambda_j))$, $1 \le j \le t$, are the corresponding spectral pairs, we denote

$$X = (X(\lambda_1),\dots,X(\lambda_t)),\ J = \mathrm{diag}(J(\lambda_1),\dots,J(\lambda_t)).$$

Then the matrix X is called a Jordan sequence of $L(\lambda)$ in $\Omega$ and the pair $(X,J)$ is the spectral pair of $L(\lambda)$ in $\Omega$. The Jordan sequence X is called regular if $X(\lambda_j)$ are regular for all $1 \leq j \leq t$. A vector $\varphi^{(0)}$ is called an eigenvector of X if $\varphi^{(0)}$ is an eigenvector of some $X(\lambda_j)$, $1 \leq j \leq t$.

Any $\lambda$-matrix $L(\lambda)$ is equivalent in $\Omega$ to a diagonal matrix

$$S(\lambda)L(\lambda)R(\lambda) = D(\lambda) = \mathrm{diag}(d_1(\lambda),d_2(\lambda),\dots,d_s(\lambda),0,\dots,0) \tag{2.1}$$

where $S(\lambda)$, $R(\lambda)$ and $D(\lambda)$ are holomorphic in $\Omega$, the matrices $S(\lambda)$ and $R(\lambda)$ are invertible and $d_{i+1}(\lambda)/d_i(\lambda)$ are holomorphic in $\Omega$ for $1 \leq i \leq s$ (see [5] for detail). If $L(\lambda)$ is regular, $s = n$, and for singular $\lambda$-matrix of order one $s = n-1$, where n is the order of the square matrix $L(\lambda)$. Let $\Omega_0$ be a bounded domain with $\bar{\Omega}_0 \subset \Omega$. Denote by $\lambda_1,\lambda_2,\dots,\lambda_t$ all the different roots of $d_s(\lambda)$ in $\Omega_0$. It may be assumed that $d_i(\lambda)$ has the form

$$d_i(\lambda) = (\lambda-\lambda_1)^{q_{i1}}(\lambda-\lambda_2)^{q_{i2}}\dots(\lambda-\lambda_t)^{q_{it}},\quad 1 \leq i \leq s, \tag{2.2}$$

where the integers $q_{ij}$ form for each $1 \leq j \leq t$ a non-decreasing sequence. If $L(\lambda)$ is regular, the number $q_j = \sum_{i=1}^{s} q_{ij}$ is a multiplicity of the eigenvalue $\lambda_j$ and is equal to a multiplicity of $\lambda_j$ as a root of the characteristic equation $|L(\lambda)| = 0$. Taking $\varphi_i(\lambda)$ equal to the i-th column of $R(\lambda)$ we conclude that $\varphi_i(\lambda)$ is a root function of multiplicity $q_{ij}$ corresponding to eigenvalue $\lambda_j$, $1 \leq j \leq t$. The root functions $\varphi_1(\lambda),\varphi_2(\lambda),\dots,\varphi_s(\lambda)$ generate for an eigenvalue $\lambda_j$

a spectral pair $(X(\lambda_j),J(\lambda_j))$, which is called a canonical spectral pair of $L(\lambda)$ corresponding to $\lambda_j$. The eigenvectors of $X(\lambda_j)$ are linear combination of $\varphi_1(\lambda_j)$, $\varphi_2(\lambda_j),\dots,\varphi_s(\lambda_j)$ - columns of the matrix $R(\lambda_j)$. If $L(\lambda)$ is singular of order one, the last column of $R(\lambda_j)$ is a singular eigenvector corresponding to $\lambda_j$. Since the columns of $R(\lambda_j)$ are independent, the sequence $X(\lambda_j)$ corresponding to the eigenvalue $\lambda_j$ is regular. Collecting all the pairs $(X(\lambda_j),J(\lambda_j))$, $1 \le j \le t$, we get the canonical spectral pair of $L(\lambda)$ in $\Omega_0$, which is denoted by $(X_{\Omega_0},J_{\Omega_0})$. The sequence $X_{\Omega_0}$ is called also a canonical Jordan sequence of $L(\lambda)$ in $\Omega_0$ and is obviously regular.

Let now $L(\lambda) = \sum\limits_{j=0}^{m} \lambda^j A_j$ be a matrix polynomial, where $A_j$ are nxn matrices. We consider first the regular case. The spectrum $\sigma(L)$ in C is finite. Taking a bounded domain $\Omega_0$ which contains the spectrum $\sigma(L)$ we consider a canonical spectral pair of $L(\lambda)$ in $\Omega_0$, which is denoted by $(X_F,J_F)$ and is called the finite canonical spectral pair of $L(\lambda)$. Similarly, $X_F$ is the finite canonical Jordan sequence. We say that $\lambda = \infty$ is an eigenvalue of $L(\lambda)$ of multiplicity [illegible] if $\lambda = 0$ is an eigenvalue of the polynomial

$$L^{(\infty)}(\lambda) = \lambda^m L(1/\lambda)$$

of the same multiplicity. Following Gohberg and Rodman in [4] we denote by $(X_\infty,J_\infty)$ a canonical spectral pair of $L^{(\infty)}(\lambda)$ corresponding to $\lambda = 0$. We call $(X_\infty,J_\infty)$ a canonical spectral pair of $L(\lambda)$ at infinity and $X_\infty$ a canonical Jordan sequence of $L(\lambda)$ at infinity. Then $X = (X_F,X_\infty)$ is called a canonical Jordan sequence of $L(\lambda)$ in infinite complex plain or simply a canonical Jordan sequence of $L(\lambda)$, and $(X,J)$, where $J = \mathrm{diag}(J_F,J_\infty)$, is a canonical spectral pair of $L(\lambda)$.

Let now $L(\lambda)$ be a singular matrix polynomial of order 1. Then the discrete spectrum $\sigma_d(L)$ of $L(\lambda)$ in C is finite. The point $\lambda = \infty$ is considered as a point of discrete spectrum of $L(\lambda)$ if $\lambda = 0$ belongs to $\sigma_d(L^{(\infty)})$. In the same way as above we define pairs $(X_F, J_F)$, $(X_\infty, J_\infty)$ and $(X, J)$. We say that the sequence $X_\infty$ is regular if it is regular with respect to $L^{(\infty)}(\lambda)$ and the eigenvalue $\lambda = 0$. Then the definition of regularity may be extended to any Jordan sequence of $L(\lambda)$ in the infinite complex plane. Obviously the canonical Jordan sequence $X$, either in the case of regular $L(\lambda)$ or in the case of singular $L(\lambda)$ of order one, is a regular Jordan sequence. If $L(\lambda)$ is singular of order one, the adjoint matrix $\mathrm{adj}L(\lambda)$ is not identically zero. Taking $\varphi_0(\lambda)$ equal to some non zero column of $\mathrm{adj}L(\lambda)$ we get a singular root function of $L(\lambda)$ which is a vector polynomial. We can assume that $\varphi_0(\lambda)$ never vanishes, otherwise the vector $\varphi_0(\lambda)$ may be reduced by a common polynomial divisor. Let $\varphi_0(\lambda)$ be vector polynomial of degree $q_0$. For any $\lambda_0$ the vectors

$$\varphi_0^{(0)}(\lambda_0), \varphi_0^{(1)}(\lambda_0), \dots, \varphi_0^{(q_0-1)}(\lambda_0), \text{ where } \varphi_0^{(j)}(\lambda_0) = \frac{1}{j!} \cdot \frac{d^j \varphi_0(\lambda)}{d\lambda^j} \bigg|_{\lambda=\lambda_0}$$

form a Jordan chain of $L(\lambda)$ corresponding to the eigenvalue $\lambda_0$. For $\lambda_0 = \infty$ the corresponding chain is defined as $\varphi_0^{(q_0-1)}(0), \varphi_0^{(q_0-2)}(0), \dots, \varphi_0^{(0)}(0)$ and is actually a Jordan chain of $L^{(\infty)}(\lambda)$ corresponding to $\lambda = 0$. The above chains are called singular Jordan chains of $L(\lambda)$ corresponding to $\lambda_0$. If $\psi_0(\lambda)$ is another singular root function of $L(\lambda)$ and $\psi_0(\lambda)$ is an irreducible vector polynomial, it is easy to show that $\psi_0(\lambda) = c\varphi_0(\lambda)$ where $c \neq 0$ is a constant.

Let $V_0$ be a space spanned by all the singular eigenvectors of $L(\lambda)$. Then $V_0$ is called the singular eigenspace of $L(\lambda)$. Since all the singular eigenvectors of $L(\lambda)$ are given by $\varphi_0(\lambda)$, we can represent

$$V_0 = \mathrm{Sp}(\varphi_0^{(0)}(\lambda_0), \varphi_0^{(1)}(\lambda_0), \dots, \varphi_0^{(q_0-1)}(\lambda_0)) \tag{2.3}$$

for any $\lambda_0 \in C$.

Finally we consider the case of a linear $\lambda$ - matrix, i.e. $L(\lambda) = A_1\lambda+A_0$. If a matrix $X_1(\lambda_0)$ is formed by the column-vectors $\varphi_1^{(0)}, \varphi_1^{(1)}, \dots, \varphi_1^{(q-1)}$ of a Jordan chain of $L(\lambda)$ and $J_1(\lambda_0)$ is the corresponding Jordan cell, we may write

$$(2.4) \qquad A_1X_1(\lambda_0)J_1(\lambda_0) + A_0X_1(\lambda_0) = 0 \, .$$

Similarly, if $(X_F, J_F)$ is some finite spectral pair of $L(\lambda)$, then

$$(2.5) \qquad A_1X_FJ_F + A_0X_F = 0 \, .$$

Since $L^{(\infty)}(\lambda) = \lambda L(1/\lambda) = A_0\lambda + A_1$, then for a spectral pair $(X_\infty, J_\infty)$ of $L(\lambda)$ at infinity we have

$$(2.6) \qquad A_1X_\infty + A_0X_\infty J_\infty = 0 \, .$$

Combining (2.5) and (2.6) we get

$$(2.7) \qquad L(\lambda)(X_F, X_\infty) = (A_1X_F, A_0X_\infty) \begin{bmatrix} \lambda-J_F & 0 \\ 0 & -\lambda J_\infty+I \end{bmatrix} .$$

In the rest of this subsection $L(\lambda)$, if not mentioned specially, is a linear singular $\lambda$-matrix of order one.

The next two lemmas follow from the canonical form of a singular pencil of matrices described in [10].

Lemma 2.1. The dimension of the singular eigenspace $V_0$ of $L(\lambda)$ is equal to $q_0$, where $q_0-1$ is the degree of the irreducible polynomial singular root function $\varphi_0(\lambda)$.

Proof: Taking $\lambda_0 = 0$ in (2.3), we note that it is enough to prove the independence of the vectors $\varphi_0^{(0)}(0), \varphi_0^{(1)}(0), \ldots, \varphi_0^{(q_0-1)}(0)$. Let us add to this chain the vector $\varphi_0^{(q_0)}(0) = 0$ and denote by $X_0$ the matrix formed by the column-vectors of the extended chain. Similarly to (2.4) we have $A_1X_0J_0 + A_0X_0 = 0$ where $J_0$ is a Jordan cell of the size $q_0+1$ with eigenvalue $\lambda = 0$. Assume that the vectors $\{\varphi_0^{(j)}(0)\}_{j=0}^{q_0-1}$ are dependent. Then there exists some vector

$$u = (u^{(0)}, u^{(1)}, \ldots, u^{(q)}, 0, \ldots, 0)' \text{ with } u^{(q)} \neq 0 \text{ and } q \leq q_0-1$$

such that $X_0u = 0$. Define a sequence of vectors $\{\psi_0^{(j)}\}_{j=0}^{q}$ by $\psi_0^{(j)} = X_0J_0^{q-j}u$. Obviously

$$\psi_0^{(0)} = u^{(q)}\varphi_0^{(0)}(0) \neq 0 \text{ and } \psi_0^{(q)} = X_0u = 0$$

Defining $\psi_0(\lambda) = \sum_{j=0}^{q} \psi_0^{(j)}\lambda^j$ we get

$$(A_1\lambda + A_0)\psi_0(\lambda) = \sum_{j=0}^{q} (A_1X_0J_0 + A_0X_0)J_0^{q-j}u\lambda^j + A_1\psi_0^{(q)}\lambda^{q+1} = 0 .$$

Therefore $\psi_0(\lambda)$ is a singular root function of $L(\lambda)$ and its degree is less than $q$, i.e. less than the degree of $\varphi_0(\lambda)$. But it was shown that $\psi_0(\lambda)$ should be proportional to $\varphi_0(\lambda)$. This contradiction proves the lemma.

Corollary 2.1. Let $\lambda_1, \lambda_2, \ldots, \lambda_t$ be distinct complex numbers (including $\lambda = \infty$) For any above $\lambda_i$ let us define a singular Jordan chain $\{\varphi_0^{(j)}(\lambda_i)\}_{j=0}^{q_i-1}$ such that $\sum_{i=1}^{t} q_i = q_0$. Then all $q_0$ such defined vectors form a basis of the space $V_0$.

Proof: The vectors $\varphi_0^{(j)}(\lambda_i)$ may be represented as a linear combination of the basis $\{\varphi_0^{(j)}(0)\}_{j=0}^{q_0-1}$ : $\varphi_0^{(j)}(\lambda_i) = (\varphi_0^{(0)}(0),\dots,\varphi_0^{(q_0-1)}(0))c_{ij}$, where $c_{ij} = \frac{1}{j!}\frac{d^j}{d\lambda^j}(1,\lambda,\dots,\lambda^{q_0-1})'|_{\lambda=\lambda_i}$. If $\lambda_i = \infty$ then $\varphi_0^{(j)}(\lambda_i) = \varphi_0^{(q_0-j-1)}(0)$. The columns $c_{ij}$ form a square $q_0 x q_0$ Vandermonde type matrix. It may be easily shown that such a matrix is invertible. Thus, the corollary is proved.

Lemma 2.2. Let $X = (X_F, X_\infty)$ be a regular Jordan sequence of $L(\lambda)$. Then the vectors of the sequence are independent of the singular eigenspace $V_0$.

Proof: Let $J = (J_F, J_\infty)$ be a Jordan matrix corresponding to X. We consider first the case when $\lambda = \infty \notin \sigma_d(L)$ and therefore $(X_\infty, J_\infty) = \emptyset$. From (2.3) and (2.4) we get $A_0V_0 \subset A_1V_0$. Denote by U the space of all complex vectors u such that $X_F u \in V_0$. Then for any $u \in U$ we have $A_1X_FJ_Fu = -A_0X_Fu \in A_0V_0 \subset A_1V_0$. Since $\lambda = \infty \notin \sigma_d(L)$, it follows that $\mathrm{Ker}A_1 \subset V_0$ and hence $X_FJ_Fu \in V_0$. Therefore $J_Fu \in U$ and the space U is an invariant space of $J_F$. Let $u_0 \in U$ be an eigenvector of $J_F$ corresponding to some eigenvalue $\lambda_0$. Then the vector $X_Fu_0$ is an eigenvector of the sequence $X_F$ and, hence, a regular eigenvector of $L(\lambda)$ corresponding to the eigenvalue $\lambda_0$. Since $X_Fu_0 \in V_0$ we can represent

$$X_Fu_0 = \sum_{j=0}^{q_0-1} c_j\varphi_0(\lambda_j)$$

where all $\lambda_j$ are finite and distinct. Then

$$0 = (A_1\lambda_0 + A_0)X_Fu_0 \quad = A_1 \sum_{j=1}^{q_0-1} c_j(\lambda_0-\lambda_j)\varphi_0(\lambda_j)$$

and therefore

$$\sum_{j=1}^{q_0-1} c_j(\lambda_0-\lambda_j)\varphi_0(\lambda_j) = c_{q_0}\varphi_0(\infty)\ .$$

But according to corollary 2.1 the vectors $\varphi_0(\lambda_1), \varphi_0(\lambda_2), \dots, \varphi_0(\lambda_{q_0-1}), \varphi_0(\infty)$ are independent. Therefore $X_F u_0 = c_0 \varphi_0(\lambda_0)$ and $X_F u_0$ is a singular eigenvector.

Let us consider now the case when $\lambda = \infty \in \sigma_d(L)$. Fixing some point $\lambda_0 \notin \sigma_d(L)$ we introduce a $\lambda$-matrix

$$\tilde{L}(\lambda) = (A_1\lambda_0 + A_0)\lambda + A_0 = \tilde{A}_1\lambda + \tilde{A}_0$$

and define a function $f(\lambda) = \lambda/(\lambda_0 - \lambda)$. Then

$$L(\lambda) = (1-\lambda/\lambda_0)\tilde{L}(f(\lambda))$$

and $\tilde{\varphi}_0(\lambda) = \varphi_0(f^{-1}(\lambda))$ is a singular root function of $\tilde{L}(\lambda)$. It is obvious that $\tilde{L}(\lambda)$ is singular of order one with the same singular eigenspace $V_0$ as the matrix $L(\lambda)$, but $\lambda = \infty \notin \sigma_d(\tilde{L})$. Denote

$$M = \begin{pmatrix} M_F & 0 \\ 0 & M_\infty \end{pmatrix}, \text{ where } M_F = f(J_F) = J_F(\lambda_0 I - J_F)^{-1}, \; M_\infty = (\lambda_0 J_\infty - I).$$

Then to an eigenvalue $\lambda_j$ of $J_F$ corresponds the eigenvalue $\tilde{\lambda}_j = f(\lambda_j)$ of $M_F$ and the corresponding eigenspaces of $J_F$ and $M_F$ coincide. The same result holds for $J_\infty$ and $M_\infty$, where $\tilde{\lambda}_\infty = f(\infty) = -1$. Therefore, if $u_0$ is an eigenvector of $M$, then $Xu_0$ is an eigenvector of the sequence $X$. The pair $(X,M)$ is not a spectral pair of $\tilde{L}(\lambda)$ but it satisfies the relation $\tilde{A}_1 XM + \tilde{A}_0 X = 0$. Then repeating our first proof for the matrix $\tilde{L}(\lambda)$ and the pair $(X,M)$ we arrive at some eigenvector $u_0$ of $M$ such that $Xu_0$ is a singular eigenvector of $\tilde{L}(\lambda)$ and, hence, of $L(\lambda)$. But $Xu_0$ is an eigenvector of the regular sequence $X$. Therefore the space $U = 0$, and the sequence $X$ is independent of $V_0$.

Remark 2.2. If $L(\lambda) = A_1\lambda + A_0$ is regular and $X$ is a regular Jordan sequence of $L(\lambda)$, then taking in lemma 2.2 the space $V_0 = 0$ we prove the independence of vectors of the sequence. If $(X,J)$ is a canonical pair, the number of vectors in $X$ is equal to the number (counted with multiplicities) of finite and infinite eigenvalues of $L(\lambda)$, i.e. equal to n. Therefore the vectors of a canonical sequence $X$ form a basis in $C^n$.

For a linear $\lambda$-matrix it is possible to define the concept of invariant space. The space $V \subset C^n$ is called an invariant space of $L(\lambda) = A_1\lambda+A_0$ with finite spectrum if $A_0V \subset A_1V$. Similarly, it is called an invariant space of $L(\lambda)$ with infinite spectrum if $A_1V \subset A_0V$. The direct sum of above spaces is called an invariant space of $L(\lambda)$. An invariant space is regular if it does not contain singular eigenvectors of $L(\lambda)$. An invariant space is singular if it is contained in $V_0$.

Let $V$ be a regular invariant space of $L(\lambda)$ with finite spectrum. If $\lambda_0 \notin \sigma_d(L)$ then $A_1\lambda_0 + A_0$ is an isomorphism on $V$. But $(A_1\lambda_0+A_0)V \subset A_1V$ and therefore also $A_1$ is an isomorphism on $V$. Let $X$ be a basis in $V$. Then we can represent $A_0X = -A_1XM$. Moreover, $M$ may be brought to the Jordan form, so that we can write

$$A_1XJ+ A_0X = 0.$$

But then the pair $(X,J)$ is a spectral pair of $L(\lambda)$ and the regularity of $V$ implies that also the sequence $X$ is regular. Analogously, for a regular invariant space with infinite spectrum we have a spectral pair $(X,J)$ with a regular Jordan sequence $X$ such that

$$A_1X + A_0XJ = 0.$$

For a regular invariant space $V$ with finite spectrum we define $\lambda_0$ as an eigenvalue of $L(\lambda)$ in $V$ if there is some eigenvector of $L(\lambda)$ in $V$, which corresponds to $\lambda_0$. The spectrum of $L(\lambda)$ in $V$ is then denoted by $\sigma(L,V)$ and consists of all eigenvalues $\lambda_0$. If $V$ is with infinite spectrum, then

$$\lambda_0 \in \sigma(L,V) \quad \text{iff} \quad 1/\lambda_0 \in \sigma(L^{(\infty)},V) .$$

Now lemma 2.2 may be formulated in terms of invariant spaces.

Lemma 2.3. If $V_1, V_2, \ldots, V_t$ are regular invariant spaces of $L(\lambda)$ with disjoint spectrum, then they form a direct sum $V = V_1 \oplus V_2 \oplus \ldots \oplus V_t$ which does not intersect the singular eigenspace $V_0$.

## 2.2. Linearization of λ-matrix.

We discuss here some linearization of a matrix polynomial $L(\lambda) = \sum_{j=0}^{m} A_j \lambda^j$ (for detailed description of linearization of λ-matrices see [5]). Define

$$\tilde{A}_0 = \begin{bmatrix} 0 & -I & 0 & \cdots & 0 \\ 0 & 0 & -I & \cdots & 0 \\ \vdots & & & & \\ & & & & -I \\ A_0 & A_1 & & \cdots & A_{m-1} \end{bmatrix} \; ; \; \tilde{A}_1 = \operatorname{diag}(I, I, \ldots, I, A_m) \; . \tag{2.8}$$

Then the linear λ-matrix $\tilde{L}(\lambda) = \tilde{A}_1 \lambda + \tilde{A}_0$ is called a linearization of $L(\lambda)$. If $L(\lambda)$ is of order $n$, then $\tilde{L}(\lambda)$ is of order $mn$.

Introduce matrix polynomials of order $mn$

$$F(\lambda) = \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ \lambda I & I & 0 & \cdots & 0 \\ \lambda^2 I & \lambda I & I & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ \lambda^{m-1} I & \cdots & & \lambda I & I \end{bmatrix} \quad \text{and} \quad E(\lambda) = \begin{bmatrix} B_{m-1}(\lambda) & B_{m-2}(\lambda) & \cdots & B_1(\lambda) & I \\ -I & 0 & \cdot & 0 & 0 \\ 0 & -I & \cdot & & \vdots \\ \vdots & & \ddots & & \vdots \\ 0 & \cdot & \cdots & -I & 0 \end{bmatrix} \tag{2.9}$$

where $B_1(\lambda) = A_m \cdot \lambda + A_{m-1}$ and $B_{j+1}(\lambda) = \lambda B_j(\lambda) + A_{m-j-1}$ for $1 \leq j \leq m-2$. Then the following identity holds

(2.10)
$$E(\lambda)\tilde{L}(\lambda)F(\lambda) = \begin{bmatrix} L(\lambda) & 0 \\ 0 & I_{(m-1)n} \end{bmatrix} .$$

Obviously, $E^{-1}(\lambda)$ and $F^{-1}(\lambda)$ are matrix polynomials too, so that the identity (2.10) proves the equivalence of the linear $\lambda$-matrix $\tilde{L}(\lambda)$ and the expansion $L(\lambda)\oplus I_{(m-1)n}$ of the matrix polynomial $L(\lambda)$.

The spectrum of $\tilde{L}(\lambda)$ coincides with the spectrum of $L(\lambda)$, and if $\varphi(\lambda)$ is a root function of $L(\lambda)$ of multiplicity q corresponding to an eigenvalue $\lambda_0$, then

$$\tilde{\varphi}(\lambda) = F_1(\lambda)\varphi(\lambda)$$

is a root function of $\tilde{L}(\lambda)$ of the same multiplicity and corresponding to the same eigenvalue $\lambda_0$. Here $F_1(\lambda)$ denotes the matrix of the first n columns of $F(\lambda)$:

$$F_1(\lambda) = (I,\lambda I,\ldots,\lambda^{m-1}I)' .$$

If $L(\lambda)$ is singular of order 1 and $\varphi_0(\lambda)$-corresponding singular root function, then $\tilde{\varphi}_0(\lambda) = F_1(\lambda)\varphi_0(\lambda)$ is a singular root function of the $\lambda$-matrix $\tilde{L}(\lambda)$, which like $L(\lambda)$ is singular of order one. If $\varphi_0(\lambda)$ is an irreducible vector polynomial of degree $q_0-1$, then $\tilde{\varphi}_0(\lambda)$ is irreducible too and deg $\tilde{\varphi}_0(\lambda) = q_0+m-2$. Therefore the dimension of the singular eigenspace $\tilde{V}_0$ of $\tilde{L}(\lambda)$ is equal to $q_0+m-1$.

To compare the matrices $\tilde{L}(\lambda)$ and $L(\lambda)$ at $\lambda = \infty$ we consider the matrices

$$\tilde{L}^{(\infty)}(\lambda) = \lambda\tilde{L}(1/\lambda) \text{ and } L^{(\infty)}(\lambda) = \lambda^m L(1/\lambda) .$$

Define $F^{(\infty)}(\lambda) = (F(\lambda))'$ and

$$(2.11)\quad E^{(\infty)}(\lambda) = \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & & & \vdots \\ C_0(\lambda) & C_1(\lambda) & \cdots\ C_{m-1}(\lambda) & I \end{bmatrix}$$

where $C_0(\lambda) = -\lambda A_0$ and $C_{j+1}(\lambda) = C_j(\lambda) - \lambda A_{j+1}$ for $0 \leq j \leq m-2$. Then the following equivalence holds

$$(2.12)\qquad E^{(\infty)}(\lambda)\tilde{L}^{(\infty)}(\lambda)F^{(\infty)}(\lambda) = I_{(m-1)n} \oplus L^{(\infty)}(\lambda) .$$

Similarly, if $\varphi(\lambda)$ is a root function of $L^{(\infty)}(\lambda)$ of multiplicity q corresponding to an eigenvalue $\lambda_0$, then

$$\tilde{\varphi}(\lambda) = F_m^{(\infty)}(\lambda)\varphi(\lambda)$$

is a root function of $\tilde{L}^{(\infty)}(\lambda)$ of the same multiplicity corresponding to the same $\lambda_0$. Here $F_m^{(\infty)}(\lambda)$ consists of the m last columns of $F^{(\infty)}(\lambda)$ i.e.

$$F_m^{(\infty)}(\lambda) = (\lambda^{m-1}I, \lambda^{m-2}I, \ldots, I)' .$$

### 2.3. Spectral theory of linear λ-matrices.

Let $L(\lambda) = A_1\lambda + A_0$ be a regular λ-matrix. Denote by $\lambda_1, \lambda_2, \ldots, \lambda_t$ all the different finite eigenvalues of $L(\lambda)$ of multiplicities $q_1, q_2, \ldots, q_t$ and by $\lambda_\infty = \infty$ the infinite eigenvalue of multiplicity $q_\infty$. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_t$ be positive oriented disjoint Jordan contours around the points $\lambda_1, \lambda_2, \ldots, \lambda_t$ and $\Gamma_\infty$ be negative oriented one surrounding all the contours above. Denote by $\Gamma_0$ the positive oriented contour obtained from $\Gamma_\infty$ by mapping $\lambda \to 1/\lambda$. Define linear operators

$$P_j = (2\pi i)^{-1} \oint_{\Gamma_j} (A_1\lambda + A_0)^{-1} A_1 d\lambda,\ j = 1, \ldots, t,$$

$$P_\infty = (2\pi i)^{-1} \int_{\Gamma_0} (A_1+A_0\lambda)^{-1}A_0 d\lambda .$$

Using the resolvent equation

$$L^{-1}(\lambda)A_1L^{-1}(\mu) = (\mu-\lambda)^{-1}(L^{-1}(\lambda)-L^{-1}(\mu))$$

we prove by standard methods that $P_j$, $j = 1,\dots,t$, are mutually orthogonal projectors. Applying the transformation $\lambda\to 1/\lambda$ we get

$$P_\infty = -(2\pi i)^{-1} \int_{\Gamma_\infty} (A_1\lambda+A_0)^{-1}A_0\lambda^{-1}d\lambda = I + (2\pi i)^{-1} \oint_{\Gamma_\infty} (A_1\lambda+A_0)^{-1}A_1 d\lambda .$$

Therefore the sum $P_1 + P_2 +\dots+ P_t + P_\infty = I$ and $P_\infty$ is also a projector orthogonal to $P_1,\dots,P_t$.

Let $\Omega_j$ be a neighbourhood of $\lambda_j$ containing the contour $\Gamma_j$. Denote by $\Phi(\Omega_j)$ the space of vector functions $\varphi(\lambda) = (\varphi^{(1)}(\lambda),\dots,\varphi^{(n)}(\lambda))$ analytic in $\Omega_j$. Define an operator $Q_j : \Phi(\Omega_j) \longrightarrow C^n$ by

$$Q_j\varphi = (2\pi i)^{-1} \oint_{\Gamma_j} L^{-1}(\lambda)\varphi(\lambda)d\lambda .$$

Obviously, $Q_j(A_1\varphi) = P_j\varphi$ for $\varphi(\lambda) = \text{const.}$, so that $\operatorname{Im} P_j \subset \operatorname{Im} Q_j$. Let $c(\lambda)$ be a scalar function analytic in $\Omega_j$. Then in the same standard way as one proves that $P_j^2 = P_j$ we may show that

$$(2.13) \qquad Q_j(c(\lambda)A_1Q_j(\varphi)) = Q_j(c(\lambda)\varphi(\lambda)) .$$

Substituting in (2.13) $c(\lambda) = 1$ we obtain $Q_j(A_1Q_j(\varphi)) = P_j(Q_j\varphi) = Q_j(\varphi)$. Therefore $\operatorname{Im} Q_j \subset \operatorname{Im} P_j$ and finally

$$(2.14) \qquad \operatorname{Im} Q_j = \operatorname{Im} P_j .$$

Let the dimension of $\operatorname{Im} P_j$ be $d_j$. There is some $n\times d_j$ matrix function $\Psi_j(\lambda)$ analytic in $\Omega_j$, such that the columns of a matrix $X_j = Q_j(\Psi_j(\lambda))$ form a basis in $\operatorname{Im} P_j$. For any $\varphi \in \Phi(\Omega_j)$ the following identity holds

$$(2.15) \qquad A_0Q_j(\varphi(\lambda)) = (2\pi i)^{-1} \oint_{\Gamma_j} (L(\lambda)-A_1\lambda)L^{-1}(\lambda)\varphi(\lambda)d\lambda = -A_1Q_j(\lambda\varphi(\lambda)) .$$

Therefore

$$A_1 Q_j(\lambda\Psi_j(\lambda)) + A_0 X_j = 0 .$$

Using (2.13) we transform $Q_j(\lambda\Psi_j(\lambda)) = Q_j(\lambda A_1 Q_j(\Psi_j(\lambda)) = Q_j(\lambda A_1 X_j)$. Representing $Q_j(\lambda A_1 X_j) = X_j M_j$ we obtain

$$A_1 X_j M_j + A_0 X_j = 0 . \tag{2.16}$$

Similarly for $\mathrm{Im}P_\infty$ there is a basis consisting of columns of a matrix $X_\infty$ such that

$$A_1 X_\infty + A_0 X_\infty M_\infty = 0 . \tag{2.17}$$

It follows from (2.13) that

$$Q_j(\lambda^q A_1 X_j) = Q_j(\lambda^{q-1} A_1 Q_j(\lambda A_1 X_j)) = Q_j(\lambda^{q-1} A_1 X_j) M_j = \ldots = X_j M_j^q .$$

Hence

$$Q_j(A_1 X_j(\lambda-\lambda_j)^{q_j}) = X_j(M_j-\lambda_j I)^{q_j} .$$

The matrix $L^{-1}(\lambda)$ has singularity of the type $(\lambda-\lambda_j)^{-q_j}$ at the point $\lambda = \lambda_j$ .

Therefore $Q_j(A_1 X_j(\lambda-\lambda_j)^{q_j})=0$ and the matrix $M_j$ has the only eigenvalue $\lambda_j$.
Similarly, the matrix $M_\infty$ has the only eigenvalue $\lambda_0 = 0$.

Denote

$$X_F = (X_1, X_2, \ldots, X_t),\ X = (X_F, X_\infty),\ T = (A_1 X_F,\ A_0 X_\infty),\ M_F = \mathrm{diag}(M_1, \ldots, M_t).$$

Then (2.16) and (2.17) may be written as

$$L(\lambda)X = T\begin{pmatrix} \lambda-M_F & 0 \\ 0 & -\lambda M_\infty + I \end{pmatrix} . \tag{2.18}$$

Since the space $C^n$ is a direct sum of $\operatorname{Im}P_j$, $j = 1,\dots,t,\infty$, the matrix X is invertible. For $\lambda \notin \sigma(L)$, $L(\lambda)$ is also invertible and so is T. Then for the determinant of $L(\lambda)$ we get $|L(\lambda)| = \text{const.}\cdot|\lambda-M_1|\;|\lambda-M_2|\;\dots\;|\lambda-M_t|$ and from the decomposition

$$|L(\lambda)| = \text{const.}\cdot(\lambda-\lambda_1)^{q_1}\cdot(\lambda-\lambda_2)^{q_2}\dots(\lambda-\lambda_t)^{q_t}$$

it follows that $|\lambda-M_j| = \text{const.}\ (\lambda-\lambda_j)^{q_j}$ for $j = 1,\dots,t$, and therefore

$$d_j = q_j\ .$$

Then we have also

$$d_\infty = n - \sum_{j=1}^{t} d_j = n - \sum_{j=1}^{t} q_j = q_\infty\ .$$

Using the notion of the invariant space we conclude from (2.15) that $\operatorname{Im}Q_j$, $j = 1,\dots,t$, is an invariant space of $L(\lambda)$ with finite spectrum and $\operatorname{Im}Q_\infty$ is an invariant space with infinite spectrum.

Choosing the suitable matrices $X_j$ we can assume that the corresponding matrices $M_j$ are in a Jordan form with the eigenvalue $\lambda_j$. We may then assume that the columns of X form a canonical Jordan sequence of $L(\lambda)$ (see the proof of lemma 2.5). We need the above spectral theory in order to investigate a perturbation of a linear $\lambda$-matrix. If the matrices $A_1$ and $A_0$ depend analytically on some vector parameter s in a neighbourhood of a point $s = s_0$, then the defined above projectors $P_j$ and operators $Q_j$ depend analytically on s near the point $s_0$. If the matrix $M_j(s_0)$ is in a Jordan form, such form, generally speaking, cannot be preserved. There is a complete description of an analytic perturbation of a Jordan matrix (see [6]). If $M_j(s_0)$ is a Jordan cell, the perturbed matrix $M_j(s)$ may be written in the form

$$M_j(s) = \begin{bmatrix} e_{q_j-1}(s)+\lambda_j & 1 & 0 & & 0 \\ e_{q_j-2}(s) & \lambda_j & 1 & & \vdots \\ \vdots & & & & 1 \\ e_0(s) & 0 & \cdot & \cdots & \lambda_j \end{bmatrix} \tag{2.19}$$

or in the form

$$(2.20) \qquad M_j(s) = \begin{bmatrix} \lambda_j & 1 & 0 & & 0 \\ 0 & \lambda_j & 1 & & \\ \cdot & \cdot & \cdot & & 1 \\ e_0(s) & e_1(s) & \ldots & & e_{q_j-1}(s)+\lambda_j \end{bmatrix} .$$

Obviously

$$|\lambda - M_j(s)| = (\lambda-\lambda_j)^{q_j} - e_{q_j-1}(s)(\lambda-\lambda_j)^{q_j-1} - \ldots - e_0(s) .$$

The characteristic equation $|L(\lambda,s)| = 0$ in a neighbourhood $\lambda \in \Omega_j$ for s close enough to $s_0$ is equivalent to the equation $|\lambda - M_j(s)| = 0$. Therefore the $\lambda$-polynomial $|\lambda - M_j(s)|$ is a Weierstrass polynomial of the function $|L(\lambda,s)|$ near the point $(\lambda_j, s_0)$ (see [9] about the Weierstrass polynomials). We use this fact especially in subsection 8.2 to construct the Kreiss symmetrizer for such a matrix $M_j(s)$. We shall need also in subsection 7.1 the following

Lemma 2.4. Let $\Omega_0$ be a bounded open [illegible] domain and denote by $\Gamma$ the positive oriented boundary of $\Omega_0$. Let $\varphi(\lambda) = (\varphi^{(1)}(\lambda), \varphi^{(2)}(\lambda), \ldots, \varphi^{(n)}(\lambda))'$ be a vector function analytic in a domain $\Omega \supset \bar{\Omega}_0$, and $L(\lambda) = A_1\lambda + A_0$ a linear regular $\lambda$-matrix such that $\sigma(L) \cap (\Omega \setminus \Omega_0) = \emptyset$. Let the integral $\int_\Gamma L^{-1}(\lambda)\varphi(\lambda)d\lambda = 0$. Then the function $L^{-1}(\lambda)\varphi(\lambda)$ is analytic in $\Omega$.

Proof. Using the representation (2.18) we can express

$$L(\lambda)^{-1}\varphi(\lambda) = X \begin{pmatrix} (\lambda I - M_F)^{-1} & 0 \\ 0 & (\lambda M_\infty - I)^{-1} \end{pmatrix} T^{-1}\varphi(\lambda) .$$

Since $(\lambda M_\infty - I)^{-1}$ is analytic in $C$ and $X$ and $T$ are invertible, the lemma may be reduced to the case $L(\lambda) = \lambda I - M_F$, where $M_F$ is a Jordan matrix.

Moreover, it is enough to consider the case when $M_F$ is a Jordan cell with an eigenvalue $\lambda_0 \in \Omega_0$. Denote

$$(\lambda I-M_F)^{-1}\varphi(\lambda) = \psi(\lambda) = (\psi^{(1)}(\lambda),\ldots,\psi^{(n)}(\lambda))' .$$

Then
$$\psi^{(k)}(\lambda) = (\varphi^{(k)}(\lambda) + \psi^{(k+1)}(\lambda))(\lambda-\lambda_0)^{-1} , \quad k = 1,\ldots,n-1 ,$$

$$\psi^{(n)}(\lambda) = \varphi^{(n)}(\lambda)(\lambda-\lambda_0)^{-1}.$$

Since
$$0 = \oint_\Gamma \psi^{(n)}(\lambda)d\lambda = 2\pi i\varphi^{(n)}(\lambda_0) ,$$

it follows that $\varphi^{(n)}(\lambda)(\lambda-\lambda_0)^{-1}$ is analytic in $\Omega$. Then the analyticity of $\psi^{(k)}(\lambda)$ is proved by induction on k from $k = n$ to $k = 1$.

Let $\Omega_0,\Omega,\Gamma$ be defined as in the above lemma, but $L(\lambda)$ be a linear singular $\lambda$-matrix of order one. Let $L(\lambda)$ be factorized in $\Omega$ as $L(\lambda) = L_1(\lambda)L_2(\lambda)$ where $L_1(\lambda)$ and $L_2(\lambda)$ are $\lambda$-matrices in $\Omega$, and $L_2(\lambda)$ invertible in $\Omega\setminus\Omega_0$ and, therefore, regular. Denote by $\lambda_1,\lambda_2,\ldots,\lambda_t$ all the different eigenvalues of $L_2(\lambda)$ in $\Omega_0$ of multiplicities $q_1,q_2,\ldots,q_t$. Assume that the eigenvectors of $L_2(\lambda)$ corresponding to any $\lambda_0\in\sigma(L_2)$ are regular eigenvectors of $L(\lambda)$ corresponding to the same $\lambda_0$. Then a root function $\varphi(\lambda)$ of $L_2(\lambda)$ of multiplicity $q_0$ corresponding to an eigenvalue $\lambda_0$ is also a regular root function of $L(\lambda)$ of multiplicity at least $q_0$. It follows that a canonical spectral pair $(X_{\Omega_0},J_{\Omega_0})$ of $L_2(\lambda)$ is also a spectral pair of $L(\lambda)$, and $X_{\Omega_0}$ is a regular Jordan sequence of $L(\lambda)$. Lemma 2.2 implies that the vectors of $X_{\Omega_0}$ are independent of the singular eigenspace $V_0$ of $L(\lambda)$. Define a linear operator $Q : \mathfrak{A}(\Omega) \to C^n$ by

$$Q\varphi = (2\pi i)^{-1} \oint_\Gamma L_2^{-1}(\lambda)\varphi(\lambda)d\lambda . \qquad (2.21)$$

Lemma 2.5. The space ImQ is a regular invariant space of $L(\lambda)$ of dimension $q = q_1+q_2+\ldots+q_t$ and the above sequence $X_{\Omega_0}$ form its basis.

Proof. Using equivalence (2.1) for $L_2(\lambda)$ in $\Omega$ we get $L_2^{-1}(\lambda) = R(\lambda)D^{-1}(\lambda)S(\lambda)$ where $D(\lambda) = \mathrm{diag}(d_1(\lambda),d_2(\lambda),\ldots,d_n(\lambda))$, and $d_i(\lambda)$, $i = 1,\ldots,n$, have a form given in (2.2). Replace the operator Q by $Q_1 = Q \cdot S^{-1}$. Since $S^{-1}$ is an isomorphism on $\Phi(\Omega_j)$, the space $\mathrm{Im}Q_1$ coincides with ImQ. For any $\lambda_j$, $j = 1,\ldots,t$, define vector functions $\psi_{ij}^{(k)}(\lambda) = D_i(\lambda-\lambda_j)^{-k-1}$ for $k = 0,1,\ldots,q_{ij}-1$, where $D_i$ is the i-th column of D. Then $Q_1\psi_{ij}^{(k)}(\lambda) = \frac{1}{k!} \frac{d^k R_i(\lambda)}{d\lambda^k}\Big|_{\lambda=\lambda_j} = \varphi_i^{(k)}(\lambda_j)$ and the vectors $\{\varphi_i^{(k)}(\lambda_j)\}_{k=0}^{q_{ij}-1}$ form a Jordan chain of $L_2(\lambda)$ corresponding to a root function $\varphi_i(\lambda) = R_i(\lambda)$ (here $R_i(\lambda)$ is the i-th column of $R(\lambda)$). The above chains form for all $1 \le j \le t$ and $1 \le i \le n$ the canonical Jordan sequence $X_{\Omega_0}$ of $L_2(\lambda)$. Thus we have proved that $X_{\Omega_0}$ belongs to ImQ. On the other hand, for any scalar function $\psi(\lambda)$ analytic in $\Omega$ may be written: $\psi(\lambda)/d_i(\lambda) = \psi^{(1)}(\lambda) + \psi^{(2)}(\lambda)/d_i(\lambda)$ where $\psi^{(1)}(\lambda)$ is analytic and $\psi^{(2)}(\lambda)/d_i(\lambda)$ is a linear combination of the functions $(\lambda-\lambda_j)^{-k-1}$, $1 \le j \le t$, $0 \le k \le q_{ij}-1$. Therefore the space $\mathrm{Im}Q_1$ is spanned by the vectors of $X_{\Omega_0}$, and being independent these vectors form a basis of ImQ. The number of the vectors in $X_{\Omega_0}$ is obviously $q = q_1+q_2+\ldots+q_t$. The space ImQ has a basis, which form a regular Jordan sequence of the $\lambda$-matrix $L(\lambda)$, and therefore it is a regular invariant space of this matrix. The lemma is proved.

Remark 2.5. The above lemma may be also applied to a regular matrix $L(\lambda)$. Then there are obviously no restrictions on the right divisor $L_2(\lambda)$ of $L(\lambda)$.

## 3. The case of bounded eigenvalues

Consider the problem (1.1) in a neighbourhood $\Omega(\zeta_0')$ defined in [illegible], where $\zeta_0' = (\omega_0', s_0')$ and $\mathrm{Re}\, s_0' \geq 0$. Since the $\lambda$-matrix $L(\lambda,\zeta) = sI+A\lambda+iB(\omega)$ is homogeneous of order one, by introducing $\lambda' = \lambda/|\zeta|$ one obtains

$$(3.1) \qquad L(\lambda,\zeta) = |\zeta| L(\lambda',\zeta') .$$

We consider $L(\lambda',\zeta')$ as a $\lambda'$-matrix depending on parameter $\zeta' \in \Omega(\zeta_0')$.

In the first part of this section we investigat in general the characteristic equation (1.6) and the singular $\lambda'$-matrix $L(\lambda',\zeta')$ for $s' = 0$.

In the second part theorem 1 is proved in the neighbourhood $\Omega(\zeta_0')$ with $s_0' \neq 0$.

In the third part the results of Section 2 are used to analyze the block structure of $L(\lambda',\zeta')$ for $\zeta' \in \Omega(\zeta_0')$ when $s_0' = 0$ and to prove theorem 3.[illegible] concerning the assumption 1.2. Then theorem 1 in $\Omega(\zeta_0')$ follows quite easily.

### 3.1. Preliminary analysis of $L(\lambda',\zeta')$.

Consider the characteristic equation

$$(3.2) \qquad |L(\lambda', \zeta')| = \sum_{j=0}^{n-1} a_j(\zeta')(\lambda')^j = 0 .$$

Since $|L(\lambda',\zeta')| = 0$ for $s' = 0$, the characteristic polynomial may be written as

$$(3.3) \qquad |L(\lambda',\zeta')| = s' p_0(\lambda', i\omega', s')$$

where

$$(3.4) \qquad p_0(\lambda', i\omega', s') = |A_I|\ |A_{II}|(\lambda')^{n-1} + \text{terms of lower order in } \lambda' .$$

Since the highest term in $p_0(\lambda',\zeta')$ does not vanish, the $\lambda'$-matrix $L(\lambda',\zeta')$

has for $\zeta' \neq 0$ exactly n-1 finite eigenvalues. Considering

$$L^{(\infty)}(\lambda',\zeta') = \lambda' L(1/\lambda',\zeta') = A + \lambda'(\zeta' I + B\omega')$$

we arrive at the characteristic polynomial

$$|L^{(\infty)}(\lambda',\zeta')| = \sum_{j=0}^{n-1} a_j(\zeta')\lambda'^{n-j} = \zeta'\lambda'|A_I||A_{II}| + \text{terms of higher order in } \lambda' .$$

Therefore the λ'-matrix $L^{(\infty)}(\lambda',\zeta')$ has for $\zeta' \neq 0$ a simple eigenvalue $\lambda' = 0$ and the matrix $L(\lambda',\zeta')$ has correspondingly a simple infinite eigenvalue $\lambda' = \infty$. The eigenvector of $L(\lambda',\zeta')$, which corresponds to $\lambda' = \infty$, belongs obviously to KerA. The strict hyperbolicity implies that $|L(\lambda',\zeta')|$ is real for real λ', s' and imaginary ω'. Hence, the coefficients $a_j(\zeta')$ in (4.4) are real for s' and ω' as above, or [illegible] are real for real [illegible] and imaginary s'.

Statement 4.1. The order n of the matrices A and B [illegible] order of $A_I$ and $A_{II}$ is equal to $\ell-1 = n-\ell$. [illegible] with $Re\zeta' \neq 0$ [illegible] $\omega' = 0$ the equation

(4.5) $$|L(\lambda',\zeta')| = 0$$

has exactly (n-1)/2 roots with $Re\lambda' < 0$ and the same number of roots with $Re\lambda' > 0$.

Proof. For $\omega' = 0$ the characteristic equation (4.5) becomes $|L(\lambda',\zeta')| = |A_I\lambda'+s'I|\cdot|A_{II}\lambda'+s'I| = 0$. If $Res' > 0$, the above equation has ℓ-1 roots with $Re\lambda' < 0$ and n-ℓ roots with $Re\lambda' > 0$. Since for any real ω' and $Res' \neq 0$ equation (4.5) has no imaginary roots, the above property of the roots λ' is valid also for real $\omega' \neq 0$. The equation $|L(\lambda',i\omega',0)|$ has no imaginary roots. Indeed, if $|L(\lambda'_0,i\omega'_0,0)| = 0$ with $Re\lambda'_0 = 0$, the characteristic polynomial $|L(\lambda'_0,i\omega'_0,s')|$ has a factor [illegible] and therefore the matrix $A\lambda'_0 + iB(\omega'_0)$ has a double eigenvalue $s' = 0$. The latter is excluded by the strict hyperbolicity. As s' approaches zero in the half plane $Res' > 0$, the roots of (4.5) approach the λ-roots of $|L(\lambda',i\omega',0)|$. Therefore the polynomial $|L(\lambda',i\omega',0)|$ has ℓ-1 roots in the half plane $Re\lambda' < 0$. [illegible] plane Re $\lambda'<0$ we obtain that the same number should be $n-\ell$. Therefore $n-\ell=\ell-1$ and the statement is proved.

Consider the singular $\lambda'$-matrix $L(\lambda',\omega',0) = A\lambda' + iB(\omega')$. It follows from (3.3) and (3.4) that for any complex $\omega'$, rank $L(\lambda',\omega',0) = n-1$ for some $\lambda'$ and, hence, $L(\lambda',\omega',0)$ is singular of order one.

The following lemma describes the singular root function of $L(\lambda',\omega',0)$.

Lemma 4.1. Under assumption 1.1 there exists a singular root function $\varphi_0(\lambda',\omega')$ of $L(\lambda',\omega',0)$ such that the components of $\varphi_0(\lambda',\omega')$ are homogeneous functions of $\lambda',\omega'_1,\ldots,\omega'_n$ of some degree $q_0-1$ and $\varphi_0(\lambda',\omega')$ does not vanish for any complex $\lambda'$ and $\omega'$. The degree of $\varphi_0(\lambda',\omega')$ as a $\lambda'$-polynomial is [illegible]

Proof. According to assumption [illegible], rank $L(\lambda',\omega',0)$ [illegible] $n-1$. [illegible] matrix adj $L(\lambda',\omega',0)$ never vanishes [illegible]

[illegible]

[illegible]tant and equal to $q_0$ [illegible]

$$\varphi\ [illegible] \in [illegible]\ A,\ \varphi\ [illegible] \in [illegible]$$

The neighbourhood [illegible]

Let $\lambda_1', \lambda_2', \ldots, \lambda_r'$ [illegible]

[illegible]

[illegible]

$$(3.8) \qquad L(\lambda',\zeta')X(\zeta') = T(\zeta')\begin{pmatrix} \lambda'-M_F(\zeta') & 0 \\ 0 & -\lambda' M_\infty(\zeta')+I \end{pmatrix}$$

where

$$M_F(\zeta') = \mathrm{diag}(M_1(\zeta'),M_2(\zeta'),\ldots,M_t(\zeta')), \qquad M_\infty(\zeta') = 0$$

and the matrices $M_j(\zeta')$ are analytic in $\Omega(\zeta_0')$. It may be assumed that $M_j(\zeta_0')$ is a Jordan matrix with the eigenvalue $\lambda_j'$. The matrix $X(\zeta')$ and hence $T(\zeta')$ are invertible in $\Omega(\zeta_0')$. If $\mathrm{Re}\,\zeta_0' > 0$, there are no eigenvalues $\lambda_j'$ with $\mathrm{Re}\,\lambda_j' = 0$.

Let us consider the more difficult case of $\mathrm{Re}\,\zeta_0'=0$. Let $\mathrm{Re}\,\lambda_j'=0$. Then there is only one eigenvector of $L(\lambda',\zeta_0')$ corresponding to $\lambda_j'$. Therefore $M_j(\zeta_0')$ is a single Jordan cell of order $q_j$. For convenience we replace $q_j$ by $q$. The perturbed matrix $M_j(\zeta')$ may be written in a form

$$M_j(\zeta') = \lambda_j' I + [illegible] + E_j(\zeta')$$

where

[illegible]

$E_j(\zeta_0') = 0$.

[illegible]

[illegible]

[illegible]

The characteristic equation (3.2) near the point $\lambda_j', \zeta_0'$ may be written as

$$(3.10) \qquad |(s'/i)I + A(\lambda'/i) + B(\omega')| =$$

$$= (s'/i - s_1'(\lambda'/i,\omega'))(s'/i - s_2'(\lambda'/i,\omega'))\cdot\ldots\cdot(s'/i - s_n'(\lambda'/i,\omega')) = 0$$

where $s_1', s_2', \ldots, s_n'$ are the distinct eigenvalues of the matrix $A(\lambda'/i)+B(\omega')$ and depend analytically on $\lambda'/i$ and $\omega'$. For one of them, say $s_1'$, we have $s_1'(\lambda_j'/i, \omega_0') = s_0'/i$. Therefore equation (3.10) near the point $(\lambda_j', \zeta_0')$ is equivalent to the equation

$$(3.11) \qquad f(\lambda'/i, \zeta') = s'/i - s_1'(\lambda'/i, \omega') = 0 .$$

The function $s_1'$ is real for real $\lambda'/i$ and $\omega'$. Expending $f(\lambda'/i, \zeta')$ in a power series $f(\lambda'/i, \zeta') = \sum_{k=0}^{\infty} f_k(\zeta')\,(\lambda'/i - \lambda_j'/i)^k$ we note that for complex $s'$

$$(3.12) \operatorname{Im} f_k(\zeta') = 0 \ , \ k = 1,2,\ldots \ , \ \operatorname{Im} f_0(\zeta') = \operatorname{Im}(s'/i - s_1'(\lambda_j'/i, \omega')) = -\operatorname{Re} s' .$$

The characteristic equation for $M_j(\zeta')$

$$(3.13) \quad |(\lambda' - M_j(\zeta'))/i| = (\lambda'/i - \lambda_j'/i)^q - e_{q-1}(\zeta')(\lambda'/i - \lambda_j'/i)^{q-1} - \ldots - e_0(\zeta') = 0$$

is equivalent in some neighbourhood of the point $(\lambda_j', \zeta_0')$ to equation (3.11). It follows that $e_0(\zeta'), \ldots, e_{q-1}(\zeta')$ are the coefficients of Weierstrass polynomial corresponding to the function $f(\lambda'/i, \zeta')$. Since the coefficients $f_k(\zeta')$, $k = 0,1,\ldots$ are real for imaginary $s'$, the same property have the coefficients $e_k(\zeta')$, $k = 0,1,\ldots,q-1$. The estimate $\operatorname{Im} e_0(\zeta') \geq \delta|\operatorname{Re} s'|$ follows then from (3.12) (see for example the general lemma 8.9 in Part II).

The matrix $M_j(\zeta')$ has for $\operatorname{Re}s' > 0$ some number $\rho_j$ of eigenvalues $\lambda'_{j1},\lambda'_{j2},\dots,\lambda'_{j\rho_j}$ in the half plane $\operatorname{Re}\lambda' < 0$ and the remaining $q_j-\rho_j$ eigenvalues with $\operatorname{Re}\lambda' > 0$. This number $\rho_j$ does not depend on $\zeta'$ and is given by

$$\rho_j = \begin{cases} 1/2q_j & \text{if } q_j \text{ is even} \\ 1/2(q_j-1) & \text{if } q_j \text{ is odd and } \operatorname{Im}e_0(\zeta') > 0 \\ 1/2(q_j+1) & \text{if } q_j \text{ is odd and } \operatorname{Im}e_0(\zeta') < 0 \,. \end{cases} \tag{3.14}$$

The matrix $X_j$ is then partitioned as $X_j = (X_{I,j},X_{II,j})$, where $X_{I,j}$ denotes the first $\rho_j$ columns of $X_j$. Analogously a $q_j$-dimensional column-vector $v_j$ is partitioned as

$$v_j = (v_{I,j},v_{II,j})'$$

Kreiss in [2] has constructed a symmetrizer, i.e. a smooth Hermitian matrix $R_j(\zeta')$ defined for $\zeta' \in \Omega(\zeta'_0)$ and satisfying the inequalities

$$\operatorname{Re}(R_j(\zeta')M_j(\zeta')) \ge \delta\operatorname{Re}s'\cdot I \tag{3.15}$$

and

$$v_j^* R_j(\zeta')v_j \ge |v_{II,j}|^2 - c|v_{I,j}|^2$$

where $c > 0$ may be set as small as one wants. $R_j(\zeta')$ depends on $c$. As in [illegible] (Lemma 2.6) we introduce a matrix $U_j(\zeta')$ defined for any $\zeta' \in \Omega(\zeta'_0)$ with $\operatorname{Re}s' > 0$, which is continuous at the point $\zeta'_0$ with $U_j(\zeta'_0) = I$ and transforms the matrix $M_j(\zeta')$ to the form

$$U_j^{-1}(\zeta')M_j(\zeta')U_j(\zeta') = (C + \operatorname{diag}(\lambda'_{j1}(\zeta'),\lambda'_{j2}(\zeta'),\dots,\lambda'_{jq_j}(\zeta'),\dots)) \tag{3.16}$$

$$= \begin{pmatrix} N_{j11} & N_{j12} \\ 0 & N_{j22} \end{pmatrix}$$

where the eigenvalues λ' of $N_{11}$ satisfy $\mathrm{Re}\lambda' < 0$ and those of $N_{22}$ have $\mathrm{Re}\lambda'$ [illegible]

Let us consider the problem (1.1). Substituting in (3.8) instead of λ' the differential operator $\frac{1}{|\zeta|}\frac{d}{dx}$ we have

$$L(\frac{d}{dx},\zeta)X(\zeta') = T(\zeta')\begin{pmatrix} \frac{d}{dx} - |\zeta|M_F(\zeta') & 0 \\ 0 & |\zeta| \end{pmatrix} .$$

Let us introduce a transformation

$$v = X^{-1}(\zeta')u \ , \ G = T^{-1}(\zeta')F \ .$$

where v and G are partitioned as

$$(3.17) \qquad v = (v_F, v_\infty)', \ G = (G_F, G_\infty)', \ \ v_F = (v^{(1)}, v^{(2)}, \dots, v^{(n-1)})',$$

$$G_F = (G^{(1)}, G^{(2)}, \dots, G^{(n-1)})'$$

and $v_\infty$ and $G_\infty$ replace the notations $v^{(n)}$ and $G^{(n)}$. Then problem [illegible] new variables v and G become:

$$(A) \quad (\frac{d}{dx} - |\zeta| \ M_F(\zeta'))v_F = G_F$$

$$(3.18) \qquad (B) \qquad |\zeta|v_\infty = G_\infty$$

$$(C) \qquad BXv = BX_F v_F = g$$

We have in (3.18) (C) $BX_\infty = 0$ since $X_\infty \in \mathrm{Ker}A$.

Since the matrices $X^{-1}(\zeta')$ and $T'(\zeta')$ are bounded in $\Omega(\zeta'_j)$, estimate [illegible] in variables v and G becomes

$$\text{Re}\, s\|v(x)\|^2 + |v_F(0)|^2 \le K\left(|g|^2 + \frac{\|G(x)\|^2}{\text{Re}\, s}\right) \tag{3.19}$$

Here

$$|Au(0)| = |AXv(0)| = |AX_F\, v_F\, (0)| \sim |v_F(0)|$$

From (3.18)(B) follows that

$$\|v_\infty(x)\|^2 = \frac{1}{|\zeta|^2}\|G_\infty(x)\|^2 \le \frac{1}{|\text{Re}\, s|^2}\|G(x)\|^2$$

Therefore it is enough to prove estimate (3.19) for $v_F$, i.e.

$$\text{Re}\, s\|v_F(x)\|^2 + |v_F(0)|^2 \le K\left(|g|^2 + \frac{\|G_F(x)\|^2}{\text{Re}\, s}\right) \tag{3.20}$$

The proof of the last estimate for problem (3.18), (A), (C) is exactly as in [illegible]. To make the reference easier we present this proof here. If Re $\lambda'_j$ [illegible], we set $R_j(\zeta') = I$ and for Re $\lambda'_j < 0$, $R_j(\zeta') = -cI$.* Let us define

$$R_F(\zeta') = \text{diag}(R_1(\zeta'), R_2(\zeta'), \ldots, R_r(\zeta'))$$

The matrix $X_F$ is partitioned as $X_F = (X_I, X_{II})$, where $X_I$ consists of the matrices $X_j$ for Re $\lambda'_j \neq 0$ and matrices $X_{II,j}$ for Re $\lambda'_j = 0$. In the same way the vector $v_F$ is represented as $v_F = (v_I, v_{II})'$. Then the symmetrizer $R_F(\zeta')$ [illegible] satisfies the inequalities

$$\text{Re}(R_F(\zeta')M_F(\zeta')) \ge \delta \cdot \text{Re}\, s' \cdot \tag{3.21}$$

---

* We should require that $M_j(\zeta'_j)$ is not exactly in the Jordan canonical form in the case of Re $\lambda'_j \neq 0$ but has the elements of the second upper diagonal replaced by a number $\varepsilon_j$, which is sufficiently small compared with Re $\lambda'_j$.

$$v_F^* R_F v_F \geq |v_{II}|^2 - c|v_I|^2$$

Applying to equation (3.18)(A) a generalized energy method as in [2] one derives an estimate

$$(3.22)\qquad \delta\cdot\mathrm{Re}\,s\|v_F(x)\|^2 + |v_{II}(0)|^2 - c|v_I(0)|^2 \leq \frac{K}{\mathrm{Re}\,s}\|G_F(x)\|^2$$

Lemma 3.3. The conditions (UKC) and $(\overline{\text{UKC}})$ in the neighbourhood $\Omega(\zeta_0')$ are equivalent to the condition

$$\det S\, X_I(\zeta_0') \neq 0$$

Proof. We complete the definition of the matrices $U_j(\zeta')$ for all $j = 1,\dots,t$ by setting $U_j(\zeta') = I$ when $\mathrm{Re}\,\lambda_j' \neq 0$. Then

$$U(\zeta') = \mathrm{diag}(U_1(\zeta'), U_2(\zeta'),\dots,U_t(\zeta'))$$

is continuous at the point $\zeta_0'$ with $U(\zeta_0') = I$. Let us introduce a new variable $y_F = U^{-1}v_F$ with partition $y_F = (y_I, y_{II})'$ as for the vector $v_F$. Consider the equations (3.18) (A), (B) with $G = 0$. Equation (3.18) (A) in the new variable becomes

$$\frac{dy_F}{dx} - |\zeta|\begin{pmatrix} N_{11} & N_{12} \\ 0 & N_{22}\end{pmatrix} y_F = 0$$

where $N_{11}$ is of order $(\ell-1)\times(\ell-1)$ with eigenvalues $\lambda'$ satisfying $\mathrm{Re}\,\lambda'$ [illegible] and $N_{22}$ has eigenvalues with $\mathrm{Re}\,\lambda' > 0$. The solution of the last equation in $L_2(R^+)$ is

$$y_{II} = 0,\quad y_I(x) = \exp(|\zeta| N_{11} [illegible]\, x) [illegible]$$

Then the general solution of the homogeneous equation [illegible]

(3.23) $\varphi(x,\zeta) = (\varphi_1(x,\zeta),\varphi_2(x,\zeta),\dots,\varphi_{\ell-1}(x,\zeta))y_I(0) = X_F(\zeta')U(\zeta')(y_I(x),0)$

so that

$$\varphi(0,\zeta) = X_F(\zeta')U(\zeta')(y_I(0),0)' .$$

The vectors $\varphi_1(0,\zeta),\dots,\varphi_{\ell-1}(0,\zeta)$ depend obviously on $\zeta'$ and are continuous functions at the point $\zeta_0'$ with the value

$$(\varphi_1(0,\zeta_0'),\dots,\varphi_{\ell-1}(0,\zeta_0')) = X_I(\zeta_0') .$$

The columns of $X_I(\zeta_0')$ are independent. In Section 1 we have defined also the condition $(\overline{\text{UKC}})$ related to the "shortened" vectors $\bar{\varphi}_j(0,\zeta_0')$ which correspond to the matrix $\bar{X}_I(\zeta_0')$. Since $\bar{X}_\infty(\zeta_0') = 0$ and the columns of $(X_I(\zeta_0'), X_\infty(\zeta_0'))$ are independent, also the columns of $\bar{X}_I(\zeta_0')$ are independent. Therefore (UKC) and $(\overline{\text{UKC}})$, are, as was stated in Section 1, equivalent in $\Omega(\zeta_0')$. According to (UKC), $\det S\ (\varphi_1(0,\zeta_0'),\dots,\varphi_{\ell-1}(0,\zeta_0')) \neq 0$ so that $\det S\ X_I(\zeta_0') \neq 0$. Thus, the lemma is proved.

Consider the boundary condition (3.18) (C)

$$SX_Fv_F(0) = SX_Iv_I(0) + SX_{II}v_{II}(0) = g .$$

Then under (UKC) we have an estimate

$$|v_I(0)|^2 \leq K(|v_{II}(0)|^2 + |g|^2) .$$

Choosing the constant c in (3.22) small enough (compared with K) one obtains at once the estimate (3.20).

To accomplish the proof of theorem 1 we should show the necessity of (UKC).

Let $\det S\ X_I(\zeta_0') = 0$ and a vector $y_I(0)$ satisfies

$$S\, X_I(\zeta_0')y_I(0) = 0 .$$

Defining a solution $\varphi(x,\zeta)$ of the homogeneous equation (1.4) by (3.23) and using the above $y_I(0)$ one obtains

$$g(\zeta') = S\varphi(0,\zeta')$$

so that $g(\zeta')$ is continuous function of $\zeta'$ at the point $\zeta' = \zeta_0'$ with

$$g(\zeta_0') = S\, X_I(\zeta_0')y_I(0) = 0 .$$

From estimate (1.2) one arrives at

$$|A\varphi(0,\zeta')|^2 \leq |g(\zeta')|^2$$

so that $A\varphi(0,\zeta_0') = 0$. But $A\varphi(0,\zeta_0') = A\, X_I(\zeta_0')y_I(0)$, and since the columns of $A\, X_I(\zeta_0')$ are independent, it follows that $y_I(0) = 0$.

Therefore det $S\, X_I(\zeta_0') \neq 0$, and (UKC) is satisfied in a sufficiently small neighbhourhood $\Omega(\zeta_0')$.

### 3.3. The neighbourhood $\Omega(\zeta_0')$ with $s_0' = 0$.

We begin with some kind of perturbation theory for the $\lambda'$-matrix $L(\lambda',\zeta')$ considered as a deformation of the singular $\lambda'$-matrix $A\lambda'+iB(\omega')$.

Let $\lambda_1',\lambda_2',\dots,\lambda_t'$ be all the different roots of the equation $p_0(\lambda',\zeta_0') = 0$ with multiplicities $q_1,q_2,\dots,q_t$. As shown in statement 3.1, exactly $(n-1)/2$ roots (counted with the multiplicities) belong to the half plane $\operatorname{Re} \lambda' < 0$ and the remaining $(n-1)/2$ roots have $\operatorname{Re} \lambda' > 0$. We add to the whole set of roots the value $\lambda_\infty' = \infty$ with multiplicity $q_\infty = 1$.

The contours $\Gamma_j$, $j = 1,\dots,t$, $\Gamma_\infty$ and $\Gamma$ are defined as in subsection 3.2 and the neighbourhood $\Omega(\zeta'_0)$ is then chosen small enough, so that for any $\zeta'\in\Omega(\zeta'_0)$ there are no roots of the equation $p_0(\lambda',\zeta') = 0$ on the above contours. For $\zeta'\in\Omega(\zeta'_0)$ with $s' \neq 0$ we define the mutually orthogonal projectors $P_j(\zeta')$, $j = 1,2,\dots,t$, and $P_\infty(\zeta')$ as in (3.6). Now these projectors are not defined for $s' = 0$. In this subsection we suppose that assumption 1.1 (but not necessarily 1.2) is satisfied. Then the following result takes place.

Lemma 3.4. For any $j = 1,2,\dots,t$ there exists an $n x q_j$ matrix valued function $X_j(\omega',s')$ analytic in $\Omega(\zeta'_0)$, which fulfils the following conditions:

a) for $s' \neq 0$ the columns of $X_j(\omega',s')$ belong to the space $\operatorname{Im} P_j(\zeta')$;

b) for $s' = 0$ these columns belong to the singular eigenspace $V_0(\omega')$ and at the point $\zeta'_0$ they form a singular Jordan chain

$$\varphi_0^{(0)}(\lambda'_j,\omega'_0),\varphi_0^{(1)}(\lambda'_j,\omega'_0),\dots,\varphi_0^{(q_j-1)}(\lambda'_j,\omega'_0)$$

where $\varphi_0(\lambda',\omega')$ is defined as in lemma 3.1;

c) there is a $q_j x q_j$ matrix-valued function $M_j(\zeta')$ analytic in $\Omega(\zeta'_0)$ such that $M_j(\zeta'_0)$ is a Jordan cell with the eigenvalue $\lambda'_j$ and

$$(3.24)\qquad A\,X_j(\zeta')M_j(\zeta') + (s'I + iB(\omega'))X_j(\zeta') = 0 \text{ for any } \zeta'\in\Omega(\zeta'_0)$$

Proof: Denote by $\Omega(\lambda'_j)$ some circular neighbourhood of the point $\lambda'_j$ [illegible] the contour $\Gamma_j$ and by $\Phi(\Omega(\lambda'_j))$ the space of vector functions $\varphi(\lambda') = (\varphi^{(1)}(\lambda'),\varphi^{(2)}(\lambda'),\dots,\varphi^{(n)}(\lambda'))$ analytic in $\Omega(\lambda'_j)$. [illegible] we introduce an operator $Q_j(\zeta'):\Phi(\Omega(\lambda'_j)) \to \mathbb{C}^n$ by

$$(3.25)\qquad Q_j(\zeta')\varphi = (2\pi i)^{-1}\oint_{\Gamma_j} L^{-1}(\lambda',\zeta')\varphi(\lambda')\,d\lambda'$$

According to (2.14) $\mathrm{Im}Q_j(\zeta') = \mathrm{Im}P_j(\zeta')$ for $\zeta'\in\Omega(\zeta'_0)$ with $s'\neq 0$. Since $p_0(\lambda'_j,\omega'_0,0) = 0$, the characteristic polynomial $|L(\lambda'_j,\omega'_0,s')|$ is divisible by $(s')^2$ and the constant matrix $A\lambda'_j + iB(\omega'_0)$ has an eigenvalue $s' = 0$ of some multiplicity $\rho \geq 2$. The matrix $A\lambda' + iB(\omega')$ has only one eigenvector, namely $\varphi_0(\lambda',\omega')$, corresponding to the eigenvalue $s' = 0$. There is some nxn matrix $D(\lambda',\omega')$ analytic and invertible for $\lambda'\in\Omega(\lambda'_j)$ and $(i\omega',0)\in\Omega(\zeta'_0)$, which provides the similarity transformation

$$D^{-1}(\lambda',\omega')(A\lambda'+iB(\omega')D(\lambda',\omega') = \begin{pmatrix} N_0(\lambda',\omega') & 0 \\ 0 & N_1(\lambda',\omega') \end{pmatrix} .$$

where

$$N_0(\lambda',\omega') = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ e_0 & e_1 & e_2 & \cdots & e_{\rho-1} \end{bmatrix} . \tag{2.26}$$

$e_k = e_k(\lambda',\omega')$, $k = 0,1,\ldots,\rho-1$, are analytic functions of $\lambda',\omega'$ with $e_0(\lambda'_j,\omega'_0) = 0$ and the matrix $N_1(\lambda'_j,\omega'_0)$ is invertible. Since $|A\lambda'+iB(\omega')|$ [illegible], it follows that $e_0(\lambda',\omega')$ [illegible] 0. Hence, we may assume that the first column of the matrix $D(\lambda',\omega')$ is equal to $\varphi_0(\lambda',\omega')$. Multiplying the matrix $N_0'(\lambda',\omega')$ [illegible] on the left consecutively by the invertible matrices

$$E_1 = \begin{pmatrix} -e_1 & -e_2 & \cdots & -e_{\rho-1} & 1 \\ 1 & 0 & \cdots & \cdot & 0 \\ 0 & 1 & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & & 1 & 0 \end{pmatrix}, \quad E_2 = \mathrm{diag}\,[illegible] , \tag{2.27}$$

$$E_3 = I + \begin{bmatrix} -e & -e_2 & \cdots & -e_{\rho-1} \\ & & & \\ & & & \end{bmatrix}$$

one arrives at

$$E_3E_2E_1(N_0+s'I) = \mathrm{diag}(e_1,1,\dots,1) + O(s') .$$

Comparing the determinants $|L(\lambda',\zeta')| = s'p_0(\lambda',\zeta')$ and $|N_0+s'I| = s'(\pm e_1 \pm e_2 s' \pm \dots \pm e_{\rho-1}(s')^{\rho-1})$ we obtain that the equation $p_0(\lambda',\zeta_0') = 0$ is equivalent in $\Omega(\lambda_j')$ to the equation $e_1(\lambda',\omega_0') = 0$. Therefore $e_1(\lambda',\omega_0') = (\lambda'-\lambda_j)^{q_j} f_1(\lambda')$ with $f_1(\lambda_j') \neq 0$. Introducing finally $E_4 = \mathrm{diag}(1/f_1(\lambda'),1,\dots,1)$ we denote

$$N_0'(\lambda',\zeta') = E_4E_3E_2E_1(N_0+s'I) = \mathrm{diag}(e_1(\lambda',\omega')/f_1(\lambda'),1,\dots,1)+O(s') . \tag{3.28}$$

The matrix $(N_0'(\lambda',\zeta')^{-1}$ is analytic at the points $\lambda'\in\Gamma_j$, $\zeta'\in\Omega(\zeta_0')$ and

$$(N_0'(\lambda',\zeta_0')^{-1} = \mathrm{diag}((\lambda'-\lambda_j')^{-q_j},1,\dots,1) .$$

Let us replace the operator in (3.25) by a new one, which is denoted again by $Q_j(\zeta')$:

$$Q_j(\zeta')\varphi = (2\pi i)^{-1} \oint_{\Gamma_j} D(\lambda',\omega')[(N_0'(\lambda',\zeta'))^{-1} \oplus O_{n-\rho}]\varphi(\lambda')d\lambda' . \tag{3.29}$$

The operator $Q_j(\zeta')$ in (3.29) is analytic in $\Omega(\zeta_0')$. Since the matrices $E_k(\lambda',\zeta')$, $k = 1,2,3,4$, are invertible for $\lambda'\in\Omega(\lambda_j')$, $s' \neq 0$, the spaces $\mathrm{Im}Q_j(\zeta')$ and $\mathrm{Im}P_j(\zeta')$ still coincide for $s' \neq 0$. The matrix $D(\lambda',\omega')[N_0'(\lambda',\zeta')^{-1} \oplus O_{n-\rho}]$ multiplied on the left by $L(\lambda',\zeta')$ becomes analytic in $\Omega(\lambda_j') \times \Omega(\zeta_0')$. Therefore we have

$$L(\lambda'_j,\zeta')Q_j(\zeta')\varphi = -AQ_j(\zeta')(\lambda'-\lambda'_j)\varphi \quad . \tag{3.30}$$

Let us define vector functions

$$\psi_k(\lambda') = ((\lambda'-\lambda'_j)^{q_j-k-1},0,0,\dots,0)' \in \Phi(\Omega(\lambda'_j)) \text{ for } k = 0,1,\dots,q_j-1$$

and a matrix

$$\Psi(\lambda') = (\psi_0(\lambda'),\psi_1(\lambda'),\dots,\psi_{q_j-1}(\lambda')) \ .$$

Then the matrix $X_j(\lambda',\zeta')$ is determined by

$$X_j(\zeta') = Q_j(\zeta')\Psi \ .$$

Condition a) of the lemma is obviously fulfilled.

For $s' = 0$

$$Q_j(\zeta')\psi_k(\lambda') = (2\pi i)^{-1} \oint_{\Gamma_j} \varphi_0(\lambda',\omega')(\lambda'-\lambda'_j)^{q_j-k-1} \cdot r_1(\lambda' \text{[illegible]} \varphi_1(\lambda',\omega')d\lambda' \in \text{[illegible]}$$

and

$$Q_j(\zeta'_0)\psi_k(\lambda') = (2\pi i)^{-1} \oint_{\Gamma_j} \varphi_0(\lambda',\omega'_0)(\lambda'-\lambda'_j)^{-k-1}d\lambda' = \frac{1}{k!} \left. \frac{\partial^k\varphi_0(\lambda',\omega'_0)}{\partial\lambda'^k} \right|_{\lambda'=\lambda'_j}$$

$$= \varphi_0^{(k)}(\lambda'_j,\omega'_0) \ .$$

So condition b) is satisfied too.

Formula (3.30) implies

$$L(\lambda'_j,\zeta')Q_j(\zeta')\psi_k = -AQ_j(\zeta')\psi_{k-1} \text{ for } k = 1,\dots,q_j-1$$

and

$$L(\lambda'_j,\zeta')Q_j(\zeta')\psi_0 = -AQ_j(\zeta')((\lambda'-\lambda'_j)^{q_j}, 0,\dots,0)' \ .$$

The determinant $|N'_0(\lambda',\zeta')|$ is an analytic function of $\lambda'$ and $\zeta'$, and

$|N'_0(\lambda',\zeta'_0)| = (\lambda'-\lambda'_j)^{q_j}$. According to the [illegible] preparation theorem [illegible]

$$(\lambda'-\lambda'_j)^{q_j}\,|N'_0(\lambda',\zeta')| = f(\lambda',\zeta') + \sum_{k=0}^{q_j-1} a_k(\zeta')(\lambda'-\lambda'_j)^{k}\;\text{[illegible]} ,$$

where $f(\lambda',\zeta')$ is an analytic function of $\lambda',\zeta'$, and $a_k(\zeta')$, $k=0,\ldots,q_j-1$, are analytic in $\Omega(\zeta'_0)$ and vanish at the point $\zeta'_0$. Since the matrix $[N'(\lambda',\zeta')]^{-1}$ in (3.20) has a singularity $|N'_0(\lambda',\zeta')|^{-1}$, it follows that

$$L(\lambda'_j,\zeta')Q_j(\zeta')\psi_0 = -A\sum_{k=0}^{q_j-1} a_k(\zeta')\;\text{[illegible]}\;\psi_{q_j-1-k} .$$

Denoting

$$M_j(\zeta') = \lambda'_j I + \begin{pmatrix} \alpha_{q_j-1}(\zeta') & 1 & \cdots \\ \alpha_{q_j-2}(\zeta') & & \cdots \\ \vdots & & \vdots \\ \alpha_0(\zeta') & & \cdots \end{pmatrix}$$

we obtain (3.24). The lemma is proved.

Remark: It may be shown that the above defined vectors $Q_j(\zeta')\psi_k$, $k=0,\ldots,q_j-1$, span the space $\mathrm{Im}Q_j(\zeta')$ also for $\zeta' \neq \zeta'_0$. For $\zeta' \neq \zeta'_0$ the dimension of $\mathrm{Im}Q_j(\zeta') = \mathrm{Im}P_j(\zeta')$ is $q_j$, and therefore the columns of the matrix $X(\zeta')$ form a basis of $\mathrm{Im}P_j(\zeta')$.

Define matrices

$$X_F(\zeta') = (X_1(\zeta'),X_2(\zeta'),\ldots,X_t(\zeta')),\quad M_F(\zeta') = \mathrm{diag}(M_1(\zeta'),M_2(\zeta'),\ldots,M_t(\zeta')) .$$

The matrix $X_F(\zeta')$ is partitioned as $X_F(\zeta') = (X_I(\zeta'),X_{II}(\zeta'))$, where $X_I(\zeta')$

[illegible]

$$[illegible]$$

[illegible] may be replaced [illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

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Proof. We prove first that $\dim V_0(\omega_0') = q_0 \geq (n+1)/2$. Indeed, if $q_0 < (n+1)/2$, the columns of $(X_I(\zeta_0'), X_\infty(\zeta_0'))$ are not independent. Let $v(0) = (v_I(0), v_\infty(0))'$ be a non-zero vector such that $(X_I(\zeta_0'), X_\infty(\zeta_0'))v(0) = 0$. Then $v_I(0) \neq 0$ and

$$u(x,s') = X_I(\omega_0',s')\exp(x\, M_I(\omega_0',s'))v_I(0)/s'$$

is a non-trivial homogeneous solution of equation (1.1) (A) in $L_2(R^+)$. Note that

$$Su(0,s') = SX_I(\omega_0',s')v_I(0)/s' = S(X_I(\omega_0',s'),X_\infty(\omega_0',s'))v(0)/s'$$

because $X_\infty(\zeta') \in \mathrm{Ker}\, A$ and $SX_\infty(\zeta') = 0$. Therefore $Su(0,s')$ is bounded as $s' \to 0$. On the other hand, there is some $x > 0$ such that the function $\hat{u}(x,s') = s'u(x,s')$ is non zero for $s' = 0$. Indeed, if $\hat{u}(x,0) = 0$, then

$$\hat{u}(z,0) = X_I(\zeta_0')\exp(z M_I(\zeta_0'))v_I(0) = 0$$

for any complex $z$. The space spanned by all the vectors $v_I(z) = \exp(z M_I(\zeta_0'))v_I(0)$ is an invariant space of the matrix $M_I(\zeta')$ containing the non-zero vector $v_I(0)$. Let $v_1$ be an eigenvector of $M_I(\zeta_0')$ in the above space corresponding to an eigenvalue $\lambda_1'$. We partition it according to the matrix $M_I$ as $v_1 = (v_1, v_2, \ldots, v_1)'$. [illegible] the partial vectors of $v_1$ except $v_1$ are zero and $v_1 = (1, 0, 0, \ldots, 0)'$ [illegible]. Therefore

$$X_I(\zeta_0')v_1 = X_I(\zeta_0')v_1 = \varphi(\lambda_1', \omega_0') \neq 0.$$

[illegible] small enough.

[illegible], which implies that

$$\|u(x,s')\|^2_{R^+} \le \frac{K|\text{[illegible]}|^2}{\mathrm{Re}\,s'} \le \frac{K}{\mathrm{Re}\,s'} \quad \text{for } \mathrm{Re}\,s' > 0 .$$

Thus, the columns of $(X_I(\zeta'), X_\infty(\zeta'))$ and $(X_{II}(\zeta'), X_\infty(\zeta'))$ are independent for any $\zeta'$ in a sufficiently small neighbourhood $\Omega(\zeta_0')$. The $q_j$ independent columns of $X_j(\zeta')$ form for $s' \neq 0$ a basis of the invariant space $\mathrm{Im}\,P_j(\zeta')$. Since the $\lambda'$-matrix $L(\lambda',\zeta')$ is then regular, it follows that the matrix $X(\zeta')$ is invertible for $s' \neq 0$. Comparing the determinants $|X(\zeta')|$ and $|T(\zeta')|$ in (3.8) we conclude that also $T(\zeta')$ is invertible for $s' \neq 0$. Denote $\hat{T}^{-1}(\zeta') = s'T^{-1}(\zeta')$. The rows of the matrices $T^{-1}(\zeta')$ and $\hat{T}^{-1}(\zeta')$ are partitioned according to the columns of $X(\zeta')$ as

$$T^{-1} = (T_1^{-1}, T_2^{-1}, \dots, T_t^{-1}, T_\infty^{-1}) = (T_F^{-1}, T_\infty^{-1}) = (T_I^{-1}, T_{II}^{-1}, T_\infty^{-1})$$

and similar notations for $\hat{T}^{-1}$.

We need the following

Lemma 3.6. Let the columns of the matrix $X_j(\zeta_0')$ be independent for $j = 1,2,\dots,t$. Then the matrix valued function $\hat{T}^{-1}(\zeta')$ is analytic in $\Omega(\zeta_0')$ and the last row of the matrix $\hat{T}_j^{-1}(\zeta_0')$, $j = 1,2,\dots,t$, is non-zero.

Proof. It follows from strong hyperbolicity that

$$\|L^{-1}(\lambda',\omega',s')\| \le \frac{K}{|\mathrm{Re}\,s'|} \quad \text{for imaginary } \lambda' \text{ and real } \omega'.$$

Using (3.8) one arrives for Re $\lambda' = 0$ at

$$\|X(\zeta')\begin{pmatrix}\lambda' - M_F(\zeta') & 0 \\ 0 & I\end{pmatrix}^{-1} T^{-1}(\zeta')\| \le \frac{K}{|\mathrm{Re}\,s'|} . \tag{3.9}$$

Consider an eigenvalue $\lambda_i'$ with Re $\lambda_i' < 0$ and let us fix some constant $\lambda_0'$ with Re $\lambda_0' < 0$ and different from all the $\lambda'$ - roots of the equation $|L(\lambda',\zeta')| =$

for all $\zeta'\in\Omega(\zeta_0')$. Define a scalar function

$$\varphi_j(\lambda') = (\lambda'-\lambda_0')^{-2}\left|(\lambda'-M_F(\zeta'))/(\lambda'-\lambda_0')\right|/\left|(\lambda'-M_j(\zeta'))/(\lambda'-\lambda_0')\right| .$$

The function $\varphi_j(\lambda')$ is analytic in the half plane $\mathrm{Re}\ \lambda' \leq 0$ and tends to zero as $1/(\lambda')^2$ when $\lambda' \to \infty$. Multiplying the matrix $L^{-1}(\lambda',\zeta')$ by $\varphi_j(\lambda')$ and integrating along the imaginary axis $\lambda'$ we have from (3.32)

$$\|X(\zeta')\int_{\mathrm{Re}\ \lambda'=0}\begin{pmatrix}\lambda'-M_F(\zeta') & 0\\ 0 & 1\end{pmatrix}^{-1}\varphi_j(\lambda)d\lambda'\ T^{-1}(\zeta')\| \leq \frac{K}{|\mathrm{Re}s'|}\int_{\mathrm{Re}\ \lambda'=0}|\varphi_j(\lambda')d\lambda'|$$

$$\leq \frac{K}{|\mathrm{Re}s'|} .$$

It is easy to show that

$$\int_{\mathrm{Re}\ \lambda'=0}\begin{pmatrix}\lambda'-M_F(\zeta') & \\ & 1\end{pmatrix}^{-1}\varphi_j(\lambda')d\lambda'=\mathrm{diag}(0,0,\ldots,\varphi_j(M_j(\zeta')),0,\ldots,0)$$

and therefore

$$\|X_j(\zeta')\varphi_j(M_j(\zeta'))T_j^{-1}(\zeta')\| \leq \frac{K}{|\mathrm{Re}s'|} .$$

Since the eigenvalue $\lambda_j'$ is not a root of $\varphi_j(\lambda')$ for any $\zeta'\in\Omega(\zeta_0')$, the matrix $\varphi_j(M_j(\zeta'))$ is invertible. It follows from independence of the columns of $X_j(\zeta_0')$ that

$$\|T_j^{-1}(\zeta')\| \leq \frac{K}{|\mathrm{Re}s'|} \text{ and } \|\hat{T}_j^{-1}(\zeta')\| \leq \frac{K|s'|}{|\mathrm{Re}s'|} .$$

The matrix $\hat{T}_j^{-1}(\zeta')$ has a singularity of the type $|T(\zeta')|^{-1}$. Since $|T(\zeta')| \neq 0$ for $s' \neq 0$ and $|T(\zeta')| = 0$ for $s' = 0$, the singularity $|T(\zeta')|^{-1}$ is of the type $|s'|^{-k}$. But for real $s'$ the matrix function $\hat{T}_j^{-1}(\zeta')$ is bounded. The analyticity of $\hat{T}_j^{-1}(\zeta')$ follows now without difficulties. In the same way we prove the analyticity of $T_j^{-1}(\zeta')$ when $\mathrm{Re}\ \lambda_j' > 0$ or $j = \infty$

Let us prove the last sentence of the lemma.

The space Im $\hat{T}^{-1}(\zeta_0')$ coincides with Ker $T(\zeta_0')$. Indeed, $T(\zeta_0')\hat{T}^{-1}(\zeta_0') = 0$ and therefore Im $\hat{T}^{-1}(\zeta_0') \subset$ Ker $T(\zeta_0')$. Conversely, if $T(\zeta_0')v = 0$ then $T(\omega_0',\varepsilon')v = \varepsilon' u(\varepsilon')$, where the vector function $u(\varepsilon')$ is analytic. Then $\hat{T}^{-1}(\omega_0',\varepsilon')u(\varepsilon') = v$ and by continuity $\hat{T}^{-1}(\omega_0',0)u(0) = v$ so that Im $\hat{T}^{-1}(\zeta_0') \supset$ Ker $T(\zeta_0')$.

For any $\lambda'$ different from $\lambda_1',\lambda_2',\dots,\lambda_t'$ the matrix $\lambda'-M_F(\zeta_0')$ is invertible. Therefore for such $\lambda'$ identity (3.5) implies that

$$\text{Ker } T(\zeta_0') = \text{Ker } L(\lambda',\zeta_0')X(\zeta_0')\begin{pmatrix}\lambda'-M_F(\zeta_0') & 0\\ 0 & I\end{pmatrix}^{-1}.$$

Let $v \in \text{Ker}\,T(\zeta_0')$ and $u = \begin{pmatrix}\lambda'-M_F(\zeta_0') & 0\\ 0 & I\end{pmatrix}^{-1} v$. We suppose the components of the vectors u and v to be partitioned according to the columns of the matrix X, i.e. $u = (u_1,u_2,\dots,u_t,u_\infty)'$, $v = (v_1,v_2,\dots,v_t,v_\infty)'$. Since the matrix $M_F(\zeta_0')$ is block diagonal, $u_j = (\lambda'-M_j(\zeta_0'))^{-1}v_j$ for $j = 1,2,\dots,t$ and $u_\infty = v_\infty$.

The dimension $q_0$ of the space $V_0(\omega')$ satisfies $n \geq q_0 \geq q_j$ for any $j = 1,2,\dots,t$. Let us fix some j, $1 \leq j \leq t$ or $j = \infty$, and adjoin to the columns of $X_j(\zeta_0')$ additional $q_0-q_j$ columns of the matrix $X(\zeta_0')$ to form a basis of $V_0(\omega_0')$ (we add only Jordan chains). Denote such obtained basis by $Y_j(\zeta_0')$. Then for any $\lambda'$ the vector $\varphi(\lambda',\omega_0')$ may be expressed as a linear combintaion

$$\varphi_0(\lambda',\omega_0') = Y_j(\zeta_0')w(\lambda'),$$

where $w(\lambda')$ is a $q_0$-dimensional column-vector. For different $\lambda'$ the vectors $\varphi_0(\lambda',\omega_0')$ span the space $V_0(\omega_0')$ and the corresponding to them vectors $w(\lambda')$ span the $q_0$-dimensional space $\mathbb{C}^{q_0}$. Therefore we may assume that for some $\lambda'$ the component of the vector $w(\lambda')$, which corresponds to the last column of $X_j(\zeta_0')$, is non zero. We may also assume that $\lambda'$ is different from $\lambda_k'$, $k=1,\dots,t$.

Extending $w(\lambda')$ to a n-dimensional vector $u = (u_1, u_2, \dots, u_t, u_\infty)$ by adding zero components in the suitable places we obtain

$$\varphi_0(\lambda', \zeta_0') = X(\zeta_0')u$$

and the last component of $u_j$ is non-zero. Then the vector

$$v = \begin{pmatrix} \lambda' - M_F(\zeta_0') & 0 \\ 0 & 1 \end{pmatrix} u$$

belongs to Ker $T(\zeta_0')$, because

$$T(\zeta_0')v = L(\lambda', \zeta_0')\, X(\zeta_0')u = L(\lambda', \zeta_0')\varphi_0(\lambda', \zeta_0') = 0.$$

Since the matrix $M_j(\zeta_0')$ is a Jordan cell, the last component of $v_j$ is proportional to the last component of $u_j$ with the coefficient $\lambda' - \lambda_j' \neq 0$. Therefore the last component of $v_j$ is different from zero, and the lemma is proved.

In our next considerations we continue to prove simultaneously theorems 3.5 and [illegible]. Let us turn to problem 1.1. By substitution $u = X(\zeta')v$, $G = T^{-1}(\zeta')F$ this problem is brought to the form (3.18). The eigenvalues of the matrix $M_I(\zeta_0')$ belong to the half plane Re $\lambda' < 0$ and those of $M_{II}(\zeta_0')$ - to the half plane Re $\lambda' > 0$. We may even assume that

$$\text{Re } M_I(\zeta') \leq -\delta I \quad \text{and} \quad \text{Re } M_{II}(\zeta') \geq \delta I \text{ for } \zeta' \in U(\zeta_0') .$$

Defining the symmetrizer $R(\zeta') = R_I(\zeta') \oplus R_{II}(\zeta')$ with $R_I(\zeta') = -cI$, $R_{II}(\zeta') = I$ and applying to equation (3.18) (A) the generalized energy method, one obtains the estimate

$$\delta|\zeta|\,\|v_F(x)\|^2 + |v_{II}(0)|^2 - c|v_I(0)|^2 \leq \frac{K\|G_F(x)\|^2}{|\zeta|} \leq \frac{K\|F(x)\|^2}{|\zeta|\cdot|s'|^2} . \tag{3.19}$$

The initial values $v_I(0)$ and $v_{II}(0)$ are given by

$$v_{II}(0) = \frac{1}{|\zeta|}\int_0^{+\infty} \exp(-M_{II}(\zeta')x)T_{II}^{-1}(\zeta')F(x/|\zeta|)dx \tag{3.34}$$

and

$$S\,X_I(\zeta')v_I(0) + S\,X_{II}(\zeta')v_{II}(0) = g\ . \tag{3.35}$$

Consider a linear operator Q acting on the space $L_2(R^+)$ of n-dimensional vector-functions F(x) with the values in $\mathbb{C}^{(n-1)/2}$ and given by

$$QF = \int_0^{+\infty} \exp(-M_{II}(\zeta_0')x)\hat{T}_{II}^{-1}(\zeta_0')F(x)dx \tag{3.36}$$

Lemma 3.7. The image of the operator Q is the whole space $\mathbb{C}^{(n-1)/2}$.

Proof: The operator Q may be expanded on the space $D(R^+)$ of generalized vector functions dual to the space of exponentially decreasing on $R^+$ vector functions. Since $D(R^+)$ *is the closure of* $L_2(R^+)$ in the weak topology of $D(R^+)$ and Q is a continuous operator on $D(R^+)$ with a finite dimensional range, it follows that $Q(L_2^n(R^+)) = Q(D(R^+))$. Taking $F(x) = F(x_0)\cdot\delta(x-x_0)$, where $\delta(x-x_0)$ is the delta function, we obtain that $Q(D(R^+))$ is spanned by all vectors v of the form $v = \exp(-M_{II}(\zeta_0')x)G_{II}$, where $G_{II} \in \mathrm{Im}\,T_{II}^{-1}(\zeta_0')$. Therefore $Q(D(R^+))$ is the minimal invariant space of the matrix $M_{II}(\zeta_0')$ containing the space $\mathrm{Im}\ T_{II}^{-1}(\zeta_0')$. We assume the vector $G_{II}$ to be partitioned according to $M_{II}(\zeta_0')$. It follows from lemma 3.6 that for any $1 \leq j \leq t$ with $\mathrm{Re}\ \lambda_j' > 0$ there is a vector $G_{II} \in \mathrm{Im}\ T_{II}^{-1}(\zeta_0')$ with non-zero last component of the partial vector $G_j$. The matrix $M_{II}(\zeta_0')$ is in a Jordan form with the Jordan cells $M_j(\zeta_0')$. It may be easily shown that the minimal invariant space of $M_{II}(\zeta_0')$, which includes the above vector $G_{II}$, will also include the all space of eigenvectors and generalized eigenvectors of $M_{II}(\zeta_0')$ corresponding to the eigenvalue $\lambda_j'$. Taking such vectors $G_{II}$ for any j with $\mathrm{Re}\ \lambda_j' > 0$ one proves that the space $Q(D(R^+))$ contains all the vectors of $\mathbb{C}^{(n-1)/2}$.

Analogously to lemma 3.3 we have

Lemma 3.8. Let the dimension of the space $V_0(\omega')$ be $q_0 \geq (n+1)/2$. Then the conditions (UKC) and $(\overline{\text{UKC}})$ are equivalent in a sufficiently small neighbourhood $\Omega(\zeta_0')$ to the condition

$$\det S\, X_I(\zeta_0') \neq 0 .$$

Proof: The general solution $\varphi(x,\zeta)$ of the homogeneous equation (1.4) for $\zeta' \in \Omega(\zeta_0')$ is given by

$$\varphi(x,\zeta) = (\varphi_1(x,\zeta),\varphi_2(x,\zeta),\dots,\varphi_{\ell-1}(x,\zeta))v_I(0) = X_I(\zeta')\exp(|\zeta|M_I(\zeta')x)v_I(0)$$

so that

$$(\varphi_1(0,\zeta),\varphi_2(0,\zeta),\dots,\varphi_{\ell-1}(0,\zeta)) = X_I(\zeta') .$$

The columns of $X_I(\zeta')$ are analytic and independent vector functions for $\zeta' \in \Omega(\zeta_0')$. Moreover, since the columns of the matrix $(X_I(\zeta_0'),\ X_\infty(\zeta_0'))$ are independent and $\mathrm{Sp}X_\infty(\zeta') = \mathrm{Ker}\ A$, also the columns of the matrix $A\ X_I(\zeta_0')$ are independent. The last is equivalent to the independence of the columns of the "shortened" matrix $\bar{X}_I(\zeta_0')$. Now the claim of the lemma is obvious.

Lemma 3.9. Let $\dim V_0(\omega') = q_0 \geq (n+1)/2$. Consider the problem (1.1) with a boundary operator S, which is a constant $\frac{n-1}{2}$xn matrix with S (Ker A) = 0. If problem (1.1) is properly posed for $\omega' = \omega_0'$ in the sense of theorem 3.5, then $\det S\ X_I(\zeta_0') \neq 0$, i.e. the condition (UKC) is fulfilled.

Proof: If $S\ X_I(\zeta_0')v_I(0) = 0$ for some vector $v_I(0) \neq 0$, then

$$u(x,\zeta') = X_I(\zeta')\exp(|\zeta|M_I(\zeta'))v_I(0)$$

is a homogeneous solution of equation (1.1) (A) and

$$S\ u(0,\zeta') = r(\zeta'),$$

where $g(\zeta')$ is an analytic vector function of $\zeta'$ with $g(\zeta_0') = 0$. Since the columns of the matrix $A\,X_I(\zeta_0')$ are independent,

$$Au(0,\zeta_0') = A\,X_I(\zeta_0')v_I(0) \neq 0.$$

We get a contradiction with estimate (1.1), which implies that

$$|Au(0,\zeta')| \leq K|g(\zeta')|$$

for any $\zeta' = (\omega_0',s')$ with $\text{Re}\,s' > 0$.

Now we are able to complete the proof of theorem 3.5.

Let us return to formula (3.34). For a fixed $|\zeta|$ we consider $v_{II}(0)$ as a function of $\zeta'$ and define

$$\hat v_{II}(0,\zeta') = sv_{II}(0) = \int_0^{+\infty} \exp(-M_{II}(\zeta')x)\hat T_{II}^{-1}(\zeta')F(x)|\zeta|\,dx \tag{3.37}$$

The function $\hat v_{II}(0,\zeta')$ for a given $F \in L_2(R^+)$ is analytic in $\Omega(\zeta_0')$. According to lemma 3.7, for a suitable $F$ one can obtain any value of $\hat v_{II}(0,\zeta_0') \in \mathbb{C}^{(n-[illegible])}$.

Let $g$ in (3.35) be zero. According to lemma 3.9 the matrix $C\,X_I(\zeta_0')$ is invertible and $\hat v_I(0,\zeta') = s\,v_I(0)$ is also analytic in $\Omega(\zeta_0')$. Estimate (1.1) implies that

$$|Au(0)|^2 \leq \frac{K\|F(x)\|^2}{\text{Re}\,s}\ .$$

Since

$$Au(0) = (1/s)\cdot A(X_I(\zeta')\hat v_I(0,\zeta') + X_{II}(\zeta')\hat v_{II}(0,\zeta'))$$

we obtain

$$A(X_I(\zeta_0')\hat v_I(0,\zeta_0') + X_{II}(\zeta_0')\hat v_{II}(0,\zeta_0')) = 0 \text{ and } X_{II}(\zeta_0')\ \hat v_{II}(0,\zeta_0') \in \{X_I(\zeta_0'), X_{[illegible]}(\zeta_0')\}.$$

But $\hat{v}_{II}(0,\zeta'_0)$ may be any vector in $\mathbb{C}^{(n-1)/2}$ and therefore

$$Sp(X_{II}(\zeta'_0)) \subset Sp(X_I(\zeta'_0),\ X_\infty(\zeta'_0)) \text{ and } Sp\,X(\zeta'_0) = Sp(X_I(\zeta'_0),\ X_\infty(\zeta'_0)).$$

According to corollary 2.1 the n column-vectors of the matrix $X(\zeta'_0)$ span the space $V_0(\omega'_0)$. Hence $V_0(\omega'_0) = Sp(X_I(\zeta'_0), X_\infty(\zeta'_0))$ and $\dim V_0(\omega'_0) = (n+1)/2$. Theorem 3.5 is thus proved.

Let A and $B(\omega'_0)$ be symmetric matrices. By setting the boundary operator in (1.1) (B) as $S\,u(0) = u_I$ one obtains for $\omega' = \omega'_0$ a properly posed problem (see, for example, [1] p. 636). Therefore theorem 3.5 implies, indeed, that the matrices A and $B_j$, $j = 1,2,\ldots,m$ satisfy assumption 1.2.

Now let assumption 1.2 be fulfilled. The necessity of the condition (KC) in theorem 1 is already proved in lemma 3.9. To accomplish the proof of sufficiency of this condition we turn back to estimate (3.33). Since the matrix $C X_I(\zeta')$ is invertible in $\Omega(\zeta'_0)$, it follows from (3.35) that

$$|v_I(0)|^2 \le K(|v_{II}(0)|^2 + |g|^2).$$

Choosing the positive constant $\varepsilon$ in (3.33) small enough (compared with the above K) one obtains

$$\|v_F(x)\|^2 + \frac{|v_F(0)|^2}{|\zeta|} \le K\left(\frac{\|F(x)\|^2}{|s|^2} + \frac{|g|^2}{|\zeta|}\right).$$

Equation (3.18) (B) implies that

$$|v_\infty(x)|^2 \le \frac{K\|F(x)\|^2}{|s|^2}.$$

Since $|u| = |X(\zeta')v| \le K|v|$, and $|Au(0)| = |AX_F(\zeta')v_F(0)| \le K|v_F(0)|$, we have

$$\|u(x)\|^2 + \frac{|Au(0)|^2}{|\zeta|} \le K\left(\frac{\|F(x)\|^2}{|s|^2} + \frac{|g|^2}{|\zeta|}\right) \tag{3.38}$$ and

(3.39) $$\operatorname{Re}s\,\|u(x)\|^2 \le K\left(|g|^2 + \frac{\|F(x)\|^2}{\operatorname{Re}s}\right)$$

To prove the required estimate (1.2) it is enough to show that

(3.40) $$|Au(0)|^2 \le K\left(|g|^2 + \frac{\|F(x)\|^2}{|\zeta|}\right).$$

We consider the vector $v_{II}(x)$ as a function of $\zeta'$, where $|\zeta|$ and $F \in L_2(R^+)$ are fixed, and denote $\hat{v}_{II}(x,\zeta') = s\, v_{II}(x)$. The vector function $\hat{v}_{II}(x,\zeta')$ satisfies the equation

(3.41) $$\left(\frac{d}{dx} - |\zeta| M_{II}(\zeta')\right)\hat{v}_{II}(x,\zeta') = |\zeta|\hat{T}^{-1}_{II}(\zeta')F(x)$$

and is an analytic function of $\zeta' \in \Omega(\zeta'_0)$. Applying to (3.41) the generalized energy method with the symmetrizer $R_{II}(\zeta') = I$ we get an estimate

(3.42) $$|\zeta|\,\|\hat{v}_{II}(x,\zeta')\|^2 + |\hat{v}_{II}(0,\zeta')|^2 \le K|\zeta|\,\|F(x)\|^2,$$

where the constant K is independent of $\zeta' \in \Omega(\zeta'_0)$, $|\zeta|$ and F. Differentiating (3.41) with respect to s' one obtains in the same way

(3.43) $$\left|\frac{\partial \hat{v}_{II}(0,\zeta')}{\partial s'}\right|^2 \le K|\zeta|\left(\|F(x)\|^2 + \|\hat{v}_{II}(x,\zeta')\|^2\right) \le K|\zeta|\,\|F(x)\|^2 .$$

Denote

$$\hat{v}_I(0,\zeta') = s\,v_I(0, [illegible]$$

The vectors $\hat{v}_I(0,\zeta')$ and $\hat{v}_{II}(0,\zeta')$ are connected by the equation

(3.44) $$S\,X_I(\zeta')\,\hat{v}_I(0,\zeta') + S\,X_{II}(\zeta')\,\hat{v}_{II}(0,\zeta') = [illegible].$$

Since $S\,X_I(\zeta')$ is invertible, $\hat{v}_I(0,\zeta')$ [illegible]

$$(3.45)\quad |\hat{v}_I(0,\zeta')|^2 \le K(|\hat{v}_{II}(0,\zeta')|^2 + |s|^2|g|^2) \le K(|\zeta|\,\|F(x)\|^2 + |\zeta|^2|g|^2).$$

Differentiating (3.44) with respect to s' and using estimates (3.42) - (3.45) we get

$$(3.46)\quad \left|\frac{\partial \hat{v}_I(0,\zeta')}{\partial s'}\right|^2 \le K(|\zeta|\,\|F(x)\|^2 + |\zeta|^2|g|^2).$$

The vector function $\hat{u}(0,\zeta')$ is also analytic in $\Omega(\zeta_0')$ and satisfies

$$(3.47)\quad \left|\frac{\partial \hat{u}(0,\zeta')}{\partial s'}\right|^2 \le K(|\zeta|\,\|F(x)\|^2 + |\zeta|^2|g|^2).$$

Note, that for $\zeta' = (\omega',0) \in \Omega(\zeta_0')$, $\hat{u}(0,\zeta') \in V_0(\omega')$ and $S\hat{u}(0,\zeta') = sg|_{s=0} = 0$.

The operator S is a monomorphism on the $(n-1)/2$ dimensional space $Sp(X_I(\omega',0))$ and $S\,X_\infty(\zeta') = 0$. Since $V_0(\omega') = Sp(X_I(\omega',0), X_\infty(\omega',0))$, it follows that $(\operatorname{Ker} S)\cap V_0(\omega') = \operatorname{Ker} A$, and therefore $A\hat{u}(0,\omega',0) = 0$.

For any $\zeta' = (\omega',s')\in \Omega(\zeta_0')$ we have

$$Au(0) = A\hat{u}(0,\zeta')/s = A(\hat{u}(0,\zeta')-\hat{u}(0,\omega',0))/(s'|\zeta|)$$

There is an estimate

$$\sup_{\zeta'\in\Omega(\zeta_0')} |(\hat{u}(0,\zeta')-\hat{u}(0,\omega',0))/s'| \le \sup_{\zeta'\in\Omega(\zeta_0')} \left|\frac{\partial \hat{u}(0,\zeta')}{\partial s'}\right|.$$

Applying (3.47) we obtain finally

$$|Au(0)|^2 \le \|A\|\cdot K(|\zeta|\,\|F(x)\|^2+|\zeta|^2|g|^2)/|\zeta|^2 \le K\left(|g|^2 + \frac{\|F(x)\|^2}{|\zeta|}\right).$$

Thus, theorem 1 is proved completely.

## 4. The case of unbounded eigenvalues

We consider problem (1.1) only in a neighbourhood $\Omega(\zeta_0')$ with $s_0' = 0$. The case $s_0' \neq 0$ does not differ from the one described in subsection 3.2. The characteristic polynomial

$$|L(\lambda',\zeta')| = \sum_{j=0}^{n-1} a_j(\zeta')(\lambda')^j = 0 \quad \text{with} \quad a_{n-1}(\zeta') = s'\cdot|A_I|\cdot|A_{II}|$$

does not vanish identically for $s' = 0$ and any real $\omega \neq 0$.

Let $a_{n-1}(\zeta_0') = a_{n-2}(\zeta_0') = \ldots = a_{n-q}(\zeta_0') = 0$, $a_{n-q-1}(\zeta_0') \neq 0$, where obviously $q \geqslant 1$. The $\lambda'$-matrix $L(\lambda',\zeta_0')$ has n-q-1 finite eigenvalues and an infinite eigenvalue $\lambda_\infty = \infty$ of multiplicitiy q+1. The characteristic polynomial of the $\lambda'$-matrix $L^{(\infty)}(\lambda',\zeta') = \lambda' L(1/\lambda',\zeta')$ is

$$|L^{(\infty)}(\lambda',\zeta')| = \sum_{j=1}^{n} a_{n-j}(\zeta')(\lambda')^j \tag{4.1}$$

and at the point $\zeta' = \zeta_0'$ it takes the form

$$|L^{(\infty)}(\lambda',\zeta_0')| = (\lambda')^{q+1}(a_{n-q-1}(\zeta_0') + \ldots + a_0(\zeta_0')(\lambda')^{n-q-1}) .$$

Since the matrix $L(\lambda',\zeta_0')$ is regular, there are matrices $X(\zeta') = (X_F(\zeta'),X_\infty(\zeta'))$ and $T(\zeta')$ analytic and invertible in $\Omega(\zeta_0')$ and also analytic matrices $M_F(\zeta')$ and $M_\infty(\zeta')$ such that (3.8) holds. However, now $M_\infty(\zeta')$ is a matrix of order $q+1 \geqslant 2$ with eigenvalues near the point $\lambda' = 0$. Since the space $\operatorname{Ker} L^{(\infty)}(\lambda' = 0, \zeta')$ : Ker A is one dimensional, the matrix $M_\infty(\zeta_0')$ may be assumed to be a Jordan cell with the eigenvalue $\lambda' = 0$.

Lemma 4.1. The matrix $M_\infty(\zeta')$, $\zeta' \in \Omega_0(\zeta_0)$ may be represented in a form

$$(4.2)\quad M_\infty(\zeta') = \left(\begin{array}{c|c} 0 & M_\infty^{(1)} \\ \hline 0 & \\ \cdot & \\ \cdot & M_\infty^{(2)}(\zeta') \\ \cdot & \\ 0 & \end{array}\right) \quad \text{with } M_\infty^{(1)} = (i,0,\ldots,0)\ ,\ M_\infty^{(2)} = iC' + iE'_\infty(\zeta')\ ,$$

where C and $E_\infty(\zeta')$ are given as in (3.9).

The coefficients $e_k(\zeta')$, $k = 0,1,\ldots,q-1$, in $E_\infty(\zeta')$ are real for imaginary $s'$ and $|\mathrm{Im}\ e_0(\zeta')| \geq \delta|\mathrm{Re}s'|$.

Proof: The matrix $M_\infty(\zeta')$, being a perturbation of the Jordan cell iC, may be written as

$$M_\infty(\zeta') = iC + iE_\infty(\zeta') = iC + i\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & e_{q-1}(\zeta') & 0 & & \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ 0 & e_1(\zeta') & 0 & & \\ e_q(\zeta') & e_0(\zeta') & 0 & \cdots & \end{pmatrix}.$$

The matrix $E_\infty(\zeta')$ satisfies a demand that on any lower diagonal there is exactly one function $e_k(\zeta')$, $k = 0,1,\ldots,q$ (see [6] for detail). The matrix $M_\infty(\zeta')$, as already mentioned, has for any $\zeta'$ an eigenvalue $\lambda' = 0$ with corresponding eigenvector belonging to Ker A. Therefore $e_q(\zeta') = 0$ and the matrix $M_\infty(\zeta')$ has the form (4.2).

For any $\zeta'\in\Omega(\zeta_0')$ and $\lambda'$ in the neighbourhood of $\lambda' = 0$, the characteristic equation

$$(4.3)\qquad |(\lambda' I - M_\infty(\zeta'))/i| = (\lambda'/i)((\lambda'/i)^{q}-(\lambda'/i)^{q-1}e_{q-1}(\zeta') \ldots -$$ [illegible]

[illegible] to the [illegible]

[illegible]

$$[illegible]$$

Comparing [illegible] and [illegible] we [illegible] coefficients of the Weierstrass [illegible] corresponding [illegible] in a neighborhood of the point [illegible] the strict hyperbolicity, [illegible] Therefore the coefficient [illegible] The zero order term [illegible] $|A_j|\cdot|A_{jj}|$, [illegible] the lemma follows as in Lemma [illegible].

The matrix $M_\infty^{(1)}$ [illegible] and the remaining [illegible] given by (3.14) where $q_j$ is replaced by [illegible] The matrix [illegible] $X_\infty = (X_\infty^{(1)}, X_\infty^{(2)})$, where $X_\infty^{(1)}$ is the first [illegible] of $X_\infty$ [illegible] for A and the matrix $X_\infty^{(2)}$ is represented as [illegible] = [illegible] consists of the first $\rho_\infty$ rows of $X_\infty$. The matrix $X_F = (X_{I,F}, X_{II,F})$ [illegible] in subsection 3.2. We also denote $X = (X_{I,F}, X_{I,\infty})$, [illegible].

Let us apply to problem (1.1) a transformation

$$[illegible]$$

where the vector functions v and [illegible] are partitioned according to the matrix X, [illegible]

$$v = (v_F, v_\infty)', \quad v_F = (v_{I,F}, v_{II,F})', \quad v_\infty = (v_\infty^{(1)}, v_\infty^{(2)})',$$

$$v_\infty^{(2)} = (v_{I,\infty}, v_{II,\infty})', \quad v_I = (v_{I,F}, v_{I,\infty})' \text{ and}$$

$$v_{II} = (v_{II,F}, v_{II,\infty})'$$

and similarly for G.

Then problem (1.1) in the new variables v and G becomes

$$
(4.6)\quad
\begin{aligned}
&(A)\quad \left(\frac{d}{dx} - |\zeta| M_F(\zeta')\right) v_F = G_F\\
&(B)\quad \left(M_\infty^{(2)}(\zeta')\frac{d}{dx} - |\zeta| I\right) v_\infty^{(2)} = G_\infty^{(2)}\\
&(C)\quad |\zeta| v_\infty^{(1)} = M_\infty^{(1)} \frac{dv_\infty^{(2)}}{dx} - G_\infty^{(1)}
\end{aligned}
$$

with the boundary condition

$$(D)\quad S\, X_I(\zeta') v_I(0) + S\, X_{II}(\zeta') v_{II}(0) = g\ .$$

For the matrices $M_F(\zeta')$ and $M_\infty^{(2)}(\zeta')$ there are Kreiss symmetrizers $R_F(\zeta')$ and $R_\infty^{(2)}(\zeta')$ such that for $\mathrm{Re}\, s' > 0$

$$
(4.7)\qquad \mathrm{Re}(R_F(\zeta')M_F(\zeta')) \geq \delta\, \mathrm{Re}\, s' I \quad , \ \mathrm{Re}(R_\infty^{(2)}(\zeta')M_\infty^{(2)}(\zeta')) \geq \delta\, \mathrm{Re}\, s' I
$$

$$v_F^* R_F(\zeta') v_F \geq |v_{II,F}|^2 - c|v_{I,F}|^2 \quad \text{and} \quad (v_\infty^{(2)})^* R_\infty^{(2)}(\zeta') v_\infty^{(2)} \geq |v_{II,\infty}|^2 - c|v_{I,\infty}|^2 .$$

Applying to equation (4.6)(A) the generalized energy method using the symmetrizer $R_F(\zeta')$ one obtains for $\mathrm{Re}\, s' > 0$

(4.8) $$\delta \mathrm{Res}\|v_F(x)\|^2 + |v_{II,F}(0)|^2 - c|v_{I,F}(0)|^2 \le \frac{K}{\mathrm{Res}}\|G_F(x)\|^2 .$$

Taking a scalar product of the equation (4.6) (B) with the vector $\frac{R_\infty^{(2)}(\zeta')}{|\zeta|}\frac{dv_\infty^{(2)}}{dx}$, integrating over $0 \le x < \infty$ and comparing real parts we have

$$\frac{1}{|\zeta|}\mathrm{Re}\langle\frac{dv_\infty^{(2)}}{dx}, R_\infty^{(2)}(\zeta')M_\infty^{(2)}(\zeta')\frac{dv_\infty^{(2)}}{dx}\rangle + (v_\infty^{(2)}(0), R_\infty^{(2)}(\zeta'(v_\infty^{(2)}(0))$$

$$= \frac{1}{|\zeta|}\mathrm{Re}\langle R_\infty^{(2)}(\zeta')\frac{dv_\infty^{(2)}}{dx}, G_\infty^{(2)}\rangle$$

where $\langle\ \rangle$ denotes the inner product in $L_2(R^+)$.

Since

$$\mathrm{Re}\, R_\infty^{(2)}(\zeta')M_\infty^{(2)}(\zeta') \ge \frac{\delta \mathrm{Res}I}{|\zeta|}$$

and

$$\|R_\infty^{(2)}(\zeta')\frac{dv_\infty^{(2)}}{dx}\|\cdot\|G_\infty^{(2)}\| \le K|\zeta|\frac{\|G_\infty^{(2)}\|^2}{\mathrm{Res}} + \frac{\delta}{2}\frac{\mathrm{Res}}{|\zeta|}\cdot\|\frac{dv_\infty^{(2)}}{dx}\|^2 .$$

we arrive at

$$\delta\frac{\mathrm{Res}}{|\zeta|^2}\|\frac{dv_\infty^{(2)}}{dx}\|^2 + |v_{II,\infty}(0)|^2 - c|v_{I,\infty}(0)|^2 \le \frac{K\|G_\infty^{(2)}\|^2}{\mathrm{Res}} .$$

It follows from (4.6) (B) and (C) that

$$\|v_\infty\|^2 \le \frac{K}{|\zeta|^2}(\|\frac{dv_\infty^{(2)}}{dx}\|^2 + \|G_\infty\|^2)$$

and therefore

(4.9) $$\delta \mathrm{Res}\|v_\infty\|^2 + |v_{II,\infty}(0)|^2 - c|v_{I,\infty}(0)|^2 \le \frac{K\|G_\infty\|^2}{\mathrm{Res}} .$$

Adding (4.8) and (4.9) we obtain finally

$$\delta \cdot \mathrm{Re}s \|v\|^2 + |v_{II}(0)|^2 - c|v_I(0)|^2 \leq \frac{K\|G\|^2}{\mathrm{Re}s}. \tag{4.10}$$

Unlike the situation in lemma 3.3 the conditions (UKC) and ($\overline{\mathrm{UKC}}$) are now, generally speaking, not equivalent. However, one can prove the following

Lemma 4.2. ($\overline{\mathrm{UKC}}$) is equivalent in $\Omega(\zeta_0')$ to the condition $\det C\, X_I(\zeta_0') \neq 0$.

Proof: There is a matrix $U_F(\zeta')$ $(\zeta' \in \Omega_0(\zeta_0'),\ \mathrm{Re}s' > 0)$ continuous at the point $\zeta_0'$ with $U_F(\zeta_0') = I$ providing a similarity transformation

$$U_F^{-1}(\zeta')M_F(\zeta')U_F(\zeta') = \begin{pmatrix} N_{11,F}(\zeta') & N_{12,F}(\zeta') \\ 0 & N_{22,F}(\zeta') \end{pmatrix}.$$

where the eigenvalues $\lambda'$ of $N_{11,F}(\zeta')$ have $\mathrm{Re}\,\lambda' < 0$ and those of $N_{22,F}(\zeta')$ have $\mathrm{Re}\,\lambda' > 0$. Similarly there is a matrix $U_\infty^{(2)}(\zeta')$ such that

$$(U_\infty^{(2)}(\zeta'))^{-1}M_\infty^{(2)}(\zeta')U_\infty^{(2)}(\zeta') = \begin{pmatrix} N_{11,\infty}(\zeta') & N_{12,\infty}(\zeta') \\ 0 & N_{22,\infty}(\zeta') \end{pmatrix}$$

and the matrices $U_\infty^{(2)}$ and $N_{ij,\infty}$ have the same features as the matrices $U_F$ and $N_{ij,F}$ respectively. Defining $U_\infty = \mathrm{diag}(I, U_\infty^{(2)})$ and $U = \mathrm{diag}(U_F, U_\infty)$ we introduce a new variable $y = U^{-1}(\zeta')v$. The vector $y$ is partitioned in the same way as the vector $v$. Equations (4.6) (A), (B), (C) with $G = 0$ are transformed to the equations

$$\text{(A)}\quad \frac{dy_F}{dx} - |\zeta| \begin{pmatrix} N_{11,F} & N_{12,F} \\ 0 & N_{22,F} \end{pmatrix} y_F = 0$$

$$\text{(4.11)(B)}\quad \begin{pmatrix} N_{11,\infty} & N_{12,\infty} \\ 0 & N_{22,\infty} \end{pmatrix} \frac{dy_\infty^{(1)}}{dx} - |\zeta| y_\infty^{(2)} = 0$$

$$\text{(C)}\quad |\zeta| y_\infty^{(1)} = M_\infty^{(1)} U_\infty^{(2)} \frac{dy_\infty^{(2)}}{dx}$$

The solution of (4.11) (A) in $L_2(R^+)$ is given by

$$y_{II,F} = 0,\ y_{I,F}(x) = \exp(|\zeta| N_{11,F}(\zeta')x)\ y_{I,F}(0).$$

Since the eigenvalues $\lambda'$ of the inverse matrices $N^{-1}_{11,\infty}$ and $N^{-1}_{22,\infty}$ have respectively Re $\lambda' < 0$ and Re $\lambda' > 0$, the solution of (4.11) (B) in $L_2(R^+)$ is given by

$$y_{II,\infty} = 0,\ y_{I,\infty}(x) = \exp(|\zeta| N^{-1}_{11,\infty}(\zeta')x) y_{I,\infty}(0).$$

Finally, the value of $y^{(1)}_{\infty}(x)$ is computed with the aid of equation (4.11) (C) so that

$$y^{(1)}_{\infty}(0) = M^{(1)}_{\infty} U^{(2)}_{\infty} (N^{-1}_{11,\infty} y_{I,\infty}(0),0)'.$$

Generally speaking, $y^{(1)}_{\infty}(0)$ is not a continuous function of $\zeta'$ for a given $y_{I,\infty}(0)$. For example, if $q = 1$ and Re $e_0(\zeta') < 0$ for Re$s' > 0$, then

$$y^{(1)}_{\infty}(0) = y_{I,\infty}(0)/e_0(\zeta') \sim 1/s'.$$

Considering a "shortened" vectors we have

(4.12) $\bar{\varphi}(x,\zeta) = (\bar{\varphi}_1(x,\zeta), \bar{\varphi}_2(x,\zeta), \ldots, \bar{\varphi}_{\ell-1}(x,\zeta)) y_I(0) = \bar{X}(\zeta')U(\zeta')(y_I(x),0)'$,

where $\varphi(x,\zeta)$ is a general solution of the homogeneous equation (1.4). The component $y^{(1)}_{\infty}$ does not participate in $\bar{\varphi}(x,\zeta)$ since the contribution of $y^{(1)}_{\infty}$ in $\varphi(x,\zeta)$ is $X^{(1)}_{\infty} y^{(1)}_{\infty} \in$ Ker A. For $x = 0$, $\bar{\varphi}(0,\zeta') = \bar{X}(\zeta')H(\zeta')(y_I(0),0)'$ and $\bar{\varphi}(0,\zeta'_0) = \bar{X}_I(\zeta'_0) y_I(0)$. The columns of $\bar{X}_I(\zeta'_0)$ are independent since the "original" columns of $X_I(\zeta'_0)$ are independent of $X^{(1)}_{\infty}(\zeta'_0) = (1,0,\ldots,0)'$. Thus the vectors $\bar{\varphi}_1(0,\zeta'), \bar{\varphi}_2(0,\zeta'), \ldots, \bar{\varphi}_{\ell-1}(0,\zeta')$ depend continuously on $\zeta'$ and satisfy the orthonormalization assumption of the definition $(\overline{\text{UKC}})$. The equality

$$S\cdot(\bar{\varphi}_1(0,\zeta_0'),\dots,\bar{\varphi}_{\ell-1}(0,\zeta_0')) = S\ \bar{X}(\zeta_0')$$

proves the lemma.

Consider the boundary condition (4.6) (D). Under $(\overline{\text{UKC}})$ we have an estimate

$$|v_I(0)|^2 \le K(|v_{II}(0)|^2 + |g|^2)\ . \tag{4.13}$$

Choosing the positive constant c in (4.10) small enough we obtain finally

$$\text{Re}s\|v\|^2 + |v_I(0)|^2 + |v_{II}(0)|^2 \le K\left(\frac{\|G\|^2}{\text{Re}s} + |g|^2\right)\ . \tag{4.14}$$

Since the norms $\|v\| = \|X^{-1}u\|$ and $\|G\| = \|T^{-1}F\|$ are correspondingly equivalent to the norms $\|u\|$ and $\|F\|$, and $|Au(0)| = |AXv(0)| = |A(X_Iv_I(0) + X_{II}v_{II}(0))|$ estimate (1.2) follows immediately from (4.14).

Let us now show that $(\overline{\text{UKC}})$ is a necessary condition in theorem 1. We define the homogeneous solutions $\varphi_1(x,\zeta),\dots,\varphi_{\ell-1}(x,\zeta)$ of equation (1.1) (A) as above. Let $S(\bar{\varphi}_1(0,\zeta_0'),\dots,\bar{\varphi}_{\ell-1}(0,\zeta_0'))y_I(0) = 0$ and consider a homogeneous solution $\varphi(x,\zeta) = (\varphi_1(x,\zeta),\varphi_2(x,\zeta),\dots,\varphi_{\ell-1}(x,\zeta))y_I(0)$. Since $S\ \varphi(0,\zeta')$ depends only on the vector $\bar{\varphi}(0,\zeta')$ and the last one is a continuous function of $\zeta'$ at the point $\zeta' = \zeta_0'$, it follows that $S\varphi(0,\zeta')$ tends to zero as $\zeta'$ tends to $\zeta_0'$. On the other hand, estimate (1.2) implies that $|A\varphi(0,\zeta')| \le K|S\varphi(0,\zeta')|$. Since the norm $|A\varphi(0,\zeta')|$ is equivalent to the norm $|\bar{\varphi}(0,\zeta')|$, it follows that $\bar{\varphi}(0,\zeta_0') = 0$ and therefore $y_I(0) = 0$. Thus, theorem 1 is proved completely.

# Part II. Difference Approximation of the Initial Boundary Value Problem

## 5. Definitions, Assumptions, Statements of Results.

### 5.1. Burstein difference approach. Definitions of stability.

Consider the initial boundary value problem (0.2) for the case of two space dimensions. Problem (0.2) is now written as

$$\text{(A)} \qquad \frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} + B\frac{\partial u}{\partial y} = F(x,y,t), \quad x \geq 0, \quad -\infty < y < \infty, \quad t \geq 0$$

$$\text{(5.1)(B)} \qquad u(x,y,0) = f(x,y)$$

$$\text{(C)} \qquad Su(0,y,t) = g(y,t)$$

The matrices $A$ and $B$ are supposed to satisfy the assumptions [illegible] and [illegible]. We approximate the differential equation (5.1) (A) by so-called Burstein difference scheme. In order to introduce this scheme we define in the space $x \geq 0$, $-\infty < y < \infty$, $t \geq 0$ a grid, which consists of points $(x_\nu, y_\mu, t_\sigma) = (\nu\Delta x, \mu\Delta y, \sigma\Delta t)$, where $\nu,\mu,\sigma$ are integers, $\nu \geq 0$, $-\infty < \mu < \infty$, $\sigma \geq 0$ and $\Delta x, \Delta y, \Delta t$ are mesh sizes in the directions $x,y,t$ respectively. We assume that $\Delta t/\Delta x$ and $\Delta t/\Delta y$ are constant. Let us denote by $E_x$ and $E_y$ the shift operators acting on the space of the grid functions $u(x,y,t)$ and given by $E_x u(x,y,t) = u(x+\Delta x,y,t)$, $E_y u(x,y,t) = u(x,y+\Delta y,t)$. Then the Burstein difference operator for the equation (5.1) (A) is written as

$$\text{(5.2)} \quad Lu(x,y,t)=u(x,y,t)-[I-2\tilde{C}(E_x,E_y)](\text{[illegible]}\, E_x^{1} \text{[illegible]} E_x^{-1} \text{[illegible]} E_y^{1} + E_y^{-1} + \text{[illegible]}\, E_x, E_y\,]\,u(x,y,t-\Delta t)$$

where
$$\tilde{C}(E_x,E_y) = \frac{1}{4i}\left[\frac{\Delta t}{\Delta x} A\cdot(E_x^{\frac{1}{2}}-E_x^{-\frac{1}{2}})(E_y^{\frac{1}{2}}+E_y^{-\frac{1}{2}}) + \frac{\Delta t}{\Delta y} B\cdot(E_x^{\frac{1}{2}}+E_x^{-\frac{1}{2}})(E_y^{\frac{1}{2}}-E_y^{-\frac{1}{2}})\right].$$

The operator $L$ includes obviously only powers $-1$, $0$ and $1$ of $E_x$ and $E_y$.

The boundary operator $S$ in (5.1) (C) is approximated by difference operator

$$Su(x,y,t) = \sum_{\sigma=0}^{s} S_\sigma(E_x,E_y)u(x,y,t-\sigma\Delta t), \quad S_\sigma(E_x,E_y) = \sum S_{\sigma,\nu,\mu} E_x^\nu E_y^\mu \tag{5.3}$$

where the sum in the expression for $S_\sigma$ is finite and includes only non-negative powers of $E_x$. We denote by $\nu_b$ the largest power of $E_x$ in all $S_\sigma$, $\sigma = 0,1,\dots,s$. Finally, the entire problem (5.1) is approximated by the difference problem

$$\begin{array}{lll} & (A) & Lu(x,y,t) = \Delta t\cdot F(x,y,t) \\ (5.4) & (B) & u(x,y,0) = f(x,y) \\ & (C) & Su(0,y,t) = g(y,t) \end{array}$$

with L and S defined in (5.2) and (5.3).

Equations (5.4) (A), (B) and (C) are considered at the grid points $(x_\nu,y_\mu,t_\sigma)$ and in equation (5.4) (A) $x = \nu\Delta x$, $t = \sigma\Delta t$ with $\nu\geq 1$, $\sigma\geq 1$, so that the operator L is defined. We assume that the matrices A and B as well as the coefficient matrices $S_{\sigma,\nu,\mu}$ are constant.

In order to give a definition of stability for the problem (5.4) we introduce norms in the corresponding spaces of grid functions. Let $\ell_2(x)$ denote the space of all grid functions $u(x_\nu)$, $x_\nu=\nu\Delta x$, $\nu\geq 0$ with $\sum_{\nu=0}^{\infty}|u(x_\nu)|^2 < \infty$ and define the scalar product $(u,v)_x = \sum_\nu (u(x_\nu),v(x_\nu))\Delta x$, where the sum goes over all grid points $x_\nu$, and norm $\|u\|_x^2 = (u,u)_x$.

Similarly we define spaces $\ell_2(y,t)$, $\ell_2(x,y)$ and $\ell_2(x,y,t)$ with scalar products and norms

$$(u,v)_{y,t} = \sum(u(y,t),v(y,t))\Delta y\Delta t \ , \quad \|u\|_{y,t}^2 = (u,u)_{y,t}$$

$$(u,v)_{x,y} = \sum(u(x,y),v(x,y))\Delta x\Delta y \ , \quad \|u\|_{x,y}^2 = (u,u)_{x,y}$$

$$(u,v)_{x,y,t} = \sum(u(x,y,t),v(x,y,t))\Delta x\Delta y\Delta t \ , \quad \|u\|_{x,y,t}^2 = (u,u)_{x,y,t}$$

The sums in the above definitions are taken over corresponding grid points.

The grid point $(x_\nu, y_\mu, t_\sigma)$ is called a boundary point if

$$0 \le \nu \le m-1\ , \quad \text{where } m = \max(\nu_b + [illegible], 2)\ . \tag{5.5}$$

The number 2 in the definition of $m$ is the maximal degree of $E_x$ in the difference operator $E_x L$ (which contains only non-negative powers of $E_x$). Given a grid function $u(x,y,t)$ we denote by $u_b(x,y,t)$ the restriction of $u(x,y,t)$ on the set of boundary points and define a norm

$$\|u_b\|^2_{y,t} = \sum (u(x,y,t), u(x,y,t))\Delta y \Delta t$$

where the sum goes over all the boundary points.
Similarly for a grid function $f(x,y)$ the restriction $f_b(x,y)$ and norm $\|f_b\|^2_{x,y}$ are defined.

As in [3] we make an assumption about solvability of the problem (5.4). Since the difference equation (5.4) (A) is explicit and provides the values of $u(x,y,t)$ for $x = x_\nu$ with $\nu \ge 1$, the solvability is equivalent to an

Assumption 5.1: The difference operator $L_0(0,E_y) = \sum$ [illegible] $E_y^{\mu}$ is an isomorphism in the space $\ell_2(y)$, i.e. the matrix $L_0([illegible], e^{i\xi})$ is invertible for any $0 \le \xi \le 2\pi$ .

Consider the difference approximation (5.[illegible]) with $g = F = 0$. We repeat definition 3.1 in [3]:

Definition 5.1. The approximation is stable if there are constants $K>0$, $\alpha_0$ such that for any $\alpha > \alpha_0$, and all $\Delta x$ and $f \in \ell_2(x,y)$ an estimate

$$\|e^{-\alpha t}u(x,y,t)\|^2_{x,y} \le K\|f(x,y)\|^2_{x,y} \tag{5.6}$$

holds for all $t = \sigma\Delta t \ge 0$.

Our next definition is a modification of [illegible]

Definition 5.1 (a). The approximation [illegible] (5.[illegible]) [illegible] an estimate

$$\|e^{-\alpha t}u(x,y,t)\|^2_{x,y} \le K(\|f(x,y)\|^2_{x,y} + \|e^{-\alpha t}u_b(x,y)\|^2_{[illegible]}) \ [illegible] \tag{5.7}$$

As in [3] the analog of Duhamel's principle gives us

Lemma 5.1. If the difference approximation is stable in the sense of (5.6) or (5.7), then for the case $f = g = 0$ the following estimates are valid correspondingly

(5.8) $$\left(\frac{\alpha-\alpha_0}{\alpha\Delta t+1}\right)^2 \|u\|^2_{\alpha,x,y,t} \le K\|F\|^2_{\alpha,x,y,t}$$

(5.9) $$\left(\frac{\alpha-\alpha_0}{\alpha\Delta t+1}\right)^2 \|u\|^2_{\alpha,x,y,t} \le K\left(\|F\|^2_{\alpha,x,y,t} + \|e^{-\alpha\Delta t}F_b\|^2_{\alpha,y,t}\right)$$

We denote here by $\|u\|_{\alpha,x,y,t}$ the norm $\|e^{-\alpha t}u\|_{x,y,t}$.

From estimate (5.8) one derives as in [3] the following estimate for the case $f = 0$, $g \neq 0$, $F \neq 0$:

(5.10) $$\left(\frac{\alpha-\alpha_0}{\alpha\Delta t+1}\right)^2 \|u\|^2_{\alpha,x,y,t} \le K\left(\|F\|^2_{\alpha,x,y,t} + \frac{1}{\Delta x}\|g\|^2_{\alpha,y,t}\right)$$

and similarly (5.9) implies

(5.11) $$\left(\frac{\alpha-\alpha_0}{\alpha\Delta t+1}\right)^2 \|u\|^2_{\alpha,x,y,t} \le K\left(\|F\|^2_{\alpha,x,y,t} + \|e^{-\alpha\Delta t}F_b\|^2_{\alpha,y,t} + \left(\frac{1}{\Delta x} + \frac{e^{-2\alpha\Delta x}}{\Delta x^2}\right)\|g\|^2_{\alpha,y,t}\right)$$

Definition 5.2. Let $f = 0$. The approximation is stable if instead of (5.6) an estimate

(5.12) $$\left(\frac{\alpha-\alpha_0}{\alpha\Delta t+1}\right)^2 \|u\|^2_{\alpha,x,y,t} \le K\left(\left(\frac{\alpha-\alpha_0}{\alpha\Delta t+1}\right)\|g\|^2_{\alpha,y,t} + \|F\|^2_{\alpha,x,y,t}\right)$$ holds.

Estimate (5.12) is obviously stronger than (5.6) but weaker than the estimate 3.7 in [3].

Correspondingly, estimate (5.11) is replaced by a stronger

Definition 5.2 (a). The approximation is stable if instead of (5.7) an estimate

(5.13) $$\left(\frac{\alpha-\alpha_0}{\alpha\Delta t+1}\right)^2 \|u\|^2_{\alpha,x,y,t} \le K\left(\left(\frac{\alpha-\alpha_0}{\alpha\Delta t+1}\right)\|g\|^2_{\alpha,y,t} + \|F\|^2_{\alpha,x,y,t} + \|e^{-\alpha\Delta t}F_b\|^2_{\alpha,y,t}\right)$$

holds.

## [illegible]. Laplace-Fourier transform of the difference approximation.

Consider the problem [illegible] with f = 0. Let us apply to this problem a Fourier transform in y with dual real variable $\xi/\Delta y$ and Laplace-Fourier transform in t with dual variable [illegible]. Let us denote by [illegible] the transforms of u,F,g and by z the expression [illegible]. The difference approximation is now reduced to a one dimensional difference problem depending on the parameters $\xi$ and z:

$$(5.14)\quad (A)\qquad L(E_x,\xi,z)u(x) = F(x)$$

$$(B)\qquad [illegible](E_x,\xi,z)u(0) = g$$

where

$$(5.15)\qquad L(E_x,\xi,z)=[illegible]z)+E_x^{-1}[illegible]2z)\cdot C(E_x,\xi)[illegible]E_x+[illegible]-[illegible]E_x[illegible]$$

$$C(E_x,\xi)=(\Delta t/\Delta x)\cdot A\cos(\xi/2)(E_x-1+(\Delta t/\Delta y)[illegible]E_x+[illegible]$$

and

$$(5.16)\qquad [illegible](E_x,\xi,z) = \sum_{[illegible]} z^{-[illegible]}[illegible]E_x,[illegible]$$

[illegible] (5.1[illegible]) and [illegible] are correspondingly [illegible]

$$(5.[illegible])\qquad \left(\frac{|z|-|z_0|}{|z|}\right)^2 [illegible]_x K\left(\Delta t\frac{|z|-|z_0|}{|z|}[illegible] + [illegible]\right)$$

$$(5.18)\qquad \left(\frac{|z|-|z_0|}{|z|}\right)^{[illegible]} [illegible]_x^2 K\left(\Delta t\frac{|z|-|z_0|}{|z|}[illegible]^2 + [illegible]_x + \frac{[illegible]}{z}[illegible]\right)$$

Here $|z_0| = e^{\alpha_0\Delta t}$ with $\alpha_0>0$, and the estimates hold for [illegible] and all $u\in\ell_2(x)$ with a positive constant K, independent of [illegible]. The disadvantage of definition [illegible] is, that it [illegible] to formulate a necessary and sufficient stability condition [illegible] in [3] can not be satisfied when characteristics [illegible]. Also theorem 1 proved in the differential case is not true for the difference approximation. Nevertheless we can derive an estimate, [illegible]

$$(5.19)\qquad \left(\frac{|z|-|z_0|}{|z|}\right)^2 \|u\|_x^2 + \frac{|z|-|z_0|}{|z|}\,\Delta t\,|P\hat{u}_b|^2 \le K\left(\Delta t\,\frac{|z|-|z_0|}{|z|}\,|g|^2 + \|F\|_x^2\right)$$

where $P$ is some linear operator, which depends on $\xi$ and $z$ and acts on the boundary values of $\hat{u}$, i.e. on the $mn$ dimensional vector

$$\hat{u}_b = (u(x_0), u(x_1), \dots, u(x_{m-1}))' .$$

The exact definition of $P$ will be given in the next sections where the problem (5.14) is studied locally for different domains of parameters $\xi$ and $z$. We shall give also locally the necessary and sufficient conditions for estimate (5.19) to hold. Since the operator $P$ depends on $\xi$ and $z$, it is impossible to formulate estimate (5.19) for the original problem (5.4).

We consider (5.17)-(5.19) as a priori estimates, i.e. $\hat{u}\in L_2(x)$ is given, and $F$ and $g$ are the values of the operators $L$ and $B$ applied to $u$. The existence of a solution $u\in L_2(x)$ for any $F\in L_2(x)$ and $g$ follows then from estimates (5.17)-(5.19).

For $|z|\to\infty$, estimates (5.17)-(5.19) take form

$$(5.20)\qquad \|u\|_x^2 \le K(\Delta t|g|^2 + \|F\|_x^2) .$$

On the other hand equations (5.14) (A) and (B) turn to the equations

$$u(x) = F(x) \quad \text{for any} \quad x = j\Delta x, \ [illegible] \quad \text{and}$$

$$B_0(E_x, e^{i\xi}, [illegible]) [illegible] = g$$

Thus estimate (5.20) is equivalent to an estimate

$$[illegible] \le K [illegible]$$

which in turn is equivalent to the solvability assumption. Therefore we shall investigate the problem (5.14) only for bounded values of z, i.e. in a compact domain of parameters $1\le|z|\le|z_\omega|$, $0\le\xi\le2\pi$. To avoid the negative power $E_x^{-1}$ in $\hat{L}(E_x,\xi,z)$ as well as to simplify the notations we replace the operator $\hat{L}(E_x,\xi,z)$ by $E_x z\hat{L}(E_x,\xi,z)$, which is denoted as before by L:

$$L(E_x,\xi,z) = (z-1)E_x + (C(E_x,\xi)/2)\cdot((E_x+1)\cos\xi/2-C(E_x,\xi)) . \quad [illegible]$$

Removing the symbol ^ from $\hat{S},\hat{u},\hat{F}$ and $\hat{g}$ we finally replace problem (5.14) by a new one

$$\text{(A)} \qquad L(E_x,\xi,z)u(x) = F(x)$$

$$(5.21) \qquad \text{(B)} \qquad S(E_x,\xi,z)u(0) = \sum_{\nu=0}^{\nu_0} S_\nu(\xi,z)u(x_\nu) = g$$

The matrices $S_\nu(\xi,z)$ in (5.21)(B) are not those $S_\nu$ appearing in (5.14). Equation (5.21) (A) differs from (5.14)(A) by factor z only. Hence, for the bounded values of z, each one of the estimates (5.17)-(5.19) holds (or does not hold) for both problems (5.14) and (5.21) simultaneously.

## [illegible] The Cauchy problem.

Let us replace in (5.2) the difference operators $E_x$ by $e^{i\varphi}$ and $E_y$ by $e^{i\xi}$. Then the amplification matrix $G(\varphi,\xi)$ is given by

$$G(\varphi,\xi) = I-2i\hat{C}(\varphi,\xi) \; [illegible] \; \varphi \; [illegible] \quad [illegible]$$

where

$$C(\varphi,\xi) = (\Delta t/\Delta x)A\sin\varphi \; [illegible] \; + (\Delta t/\Delta y)B\sin \; [illegible] \; \varphi \; [illegible]$$

It follows from strict hyperbolicity that the matrix $C(\varphi,\xi)$ is diagonalizable: $C(\varphi,\xi) \sim \operatorname{diag}(\lambda_1,\lambda_2,\dots,\lambda_n)$, where [illegible] and $\lambda_1,\lambda_2,\dots,\lambda_n$ are real, [illegible] and non-zero (except the case $\xi=\varphi=0$, [illegible], when $C(\varphi,\xi)$ [illegible]).

Let us denote $\alpha = \sin(\varphi/2)\cos(\xi/2)$, $\beta = \sin(\xi/2)\cos(\varphi/2)$. We shall choose the constants $\Delta t/\Delta x$ and $\Delta t/\Delta y$ such that the eigenvalues $\lambda_j$ of the matrix $(\Delta t/\Delta x)A\alpha + (\Delta t/\Delta y)B\beta$ will satisfy

$$(5.24) \qquad |\lambda_j| / \sqrt{\alpha^2+\beta^2} \le \Lambda_{max} < 1 .$$

From now on the fractions $\Delta t/\Delta x$ and $\Delta t/\Delta y$ will be included in $A$ and $B$ respectively. Therefore the eigenvalues $a_j$ and $b_j$ of the new matrices $A$ and $B$ satisfy

$$(5.25) \qquad |a_j| < 1, \quad |b_j| < 1 .$$

The eigenvalues $z_j$ of $G(\varphi,\xi)$ are given by

$$(5.26) \qquad z_j = 1-2i\,\lambda_j \cos\varphi/2 \cos\xi/2 - 2\lambda_j^2 .$$

Then $|z_j|^2 = 1-4\lambda_j^2(1-\cos^2(\xi/2)\cos^2(\varphi/2)-\lambda_j^2) \le 1-4\lambda_j^2(1-\cos^2(\xi/2)\cos^2(\varphi/2) - \Lambda_{max}^2(\alpha^2+\beta^2)) = 1 - 4\lambda_j^2(\sin^2(\varphi/2)\sin^2(\xi/2) + (1-\Lambda_{max}^2)(\alpha^2+\beta^2)) \le 1-4\lambda_j^2(1-\Lambda_{max}^2)(\alpha^2+\beta^2)$.

From (5.26) and the last inequality one immediately derives

Statement 5.1. The eigenvalues $z_j$, $1\le j\le n$, of the amplification matrix $G(\varphi,\xi)$ satisfy $z_1 = 1$, $|z_j|\le 1$, $j = 2,3,\dots,n$. Furthermore, if $|z_j|=1$ for $j\ge 2$, it follows that $\xi=\varphi=0,\pi$ (mod $2\pi$) and hence $z_j=1$. For $\varphi$ and $\xi$ near the point $\varphi = \xi = 0$ there is an estimate

$$(5.27) \qquad |z_j|^2 \le 1-\delta(\varphi^2+\xi^2), \quad j=2,3,\dots,n,$$

where $\delta$ is some positive constant.

Condition (5.24) is therefore sufficient for the stability of the Cauchy problem connected with equation [illegible].

## [illegible] Assumptions and Conditions.

Let us first summarize the assumptions about the matrices A and B. We consider only the case of bounded eigenvalues. The matrices A and B are supposed to fulfil the requirement of strict hyperbolicity and assumptions 1.1 and 1.2. Particularly we demand that the left upper element $b_{11}$ of the matrix B is zero (when the matrix A is written in the diagonal form $A = \mathrm{diag}(0,A_I,A_{II})$). This demand does not restrict the generality of the problem in the differential case. However, for the difference approximation in the case $b_{11} \neq 0$ there appear new difficulties, which may be resolved, but would probably double the size of this work. In Section 9, investigating problem (5.22) in a neighbourhood of the point $\xi=\pi$, $z=1$, we make additional assumptions 9.1, 9.2 and 9.3. It should be noted that if $\varphi$ and $\xi$ in (5.22) tend to $\pi$, the amplification matrix $G(\varphi,\xi)$ approximates in some sense the parabolic differential equation $\frac{\partial u}{\partial t} = (A\frac{\partial^2}{\partial y^2} + B\frac{\partial^2}{\partial x^2})u$.

Estimate (9.4) and the uniform Kreiss condition used for hyperbolic systems are not natural for parabolic ones. Assumptions 9.1–9.3 arise, naturally, because we tried to apply the concepts of hyperbolic problems to the problem of a parabolic nature. This work originated in searching for the stability of Burstein difference approximation for the acoustic equations. The matrices A and B in this case are given by

$$A = \begin{pmatrix} 0 & c & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & c & 0 \end{pmatrix},$$ and all the above assumptions are satisfied.

The difference approximation, as already noted, should satisfy the regularity assumption 5.1 and inequality ([illegible]), which insure the stability of Cauchy problem and the dissipativity of fourth order for the difference scheme, when $\varphi$ and $\xi$ are near the point (0,0). With the difference operator $L(E_x,\xi,z)$ is connected a $\kappa$-matrix $L(\kappa,\xi,z)$ [illegible]. While the common tradition in stability theory of difference approximations, we use in this part the letter $\kappa$ instead of [illegible] as a parameter in the matrix $L(\kappa,\xi,z)$ and say $\kappa$-matrix instead of [illegible]-matrix.

Consider the characteristic equation

(5.28) $$|L(\kappa,\xi,z)| = 0 .$$

For $|z| \ge 1$, $z \ne 1$ it has no eigenvalues $\kappa$ on the unit circle $|\kappa| = 1$. Equation (5.28) as $z$ tends to infinity, is equivalent to the equation $\kappa^n = 0$, and therefore has exactly $n$ roots inside the unit circle. This number of roots does not depend on $z$ in the domain $|z| \ge 1$, $z \ne 1$. Since the $\kappa$-matrix $L(\kappa,\xi,z)$ is regular, the homogeneous equation

(5.29) $$L(E_x,\xi,z)u(x) = 0 \quad x = \nu\Delta x , \ \nu = 0,1,\dots$$

has exactly $n$ independent solutions $\varphi_1(x,\xi,z)$, $\varphi_2(x,\xi,z),\dots,$ $\varphi_n(x,\xi,z)$ belonging to $\ell_2(x)$.

Let us orthonormalize these solutions on the boundary points, i.e. orthonormalize the nm-dimensional vectors
$\varphi_{j,b}(x,\xi,z) = (\varphi_j(x_0,\xi,z),\dots,\varphi_j(x_{m-1},\xi,z))'$, $j = 1,2,\dots,n$
Since $m \ge$ [illegible], i.e. $m$ is at least the degree of the $\kappa$-matrix $L(\kappa,\xi,z)$, the vectors $\varphi_{j,b}$ are independent and the above orthonormalization may be done.

Denote

(5.30) $$N(\xi,z) = B(E_x,\xi,z)[\varphi_1(0,\xi,z),\dots,\varphi_n(0,\xi,z)]$$
$$= \sum_{\nu=0}^{t} B_\nu(\xi,z)[\varphi_1(x_\nu,\xi,z),\dots,\varphi_n(x_\nu,\xi,z)] .$$

Then the uniform Kreiss condition (UKC) is formulated:
UKC. There is a constant $\delta > 0$ such that $|N(\xi,z)| \ge \delta$ for any $0 \le \xi \le \pi$ and $|z| \ge 1$, $z \ne 1$.
The UKC may be also formulated as for the differential case in terms of eigenvalues and generalized eigenvectors. However, [illegible] that in the neighbourhood of the points $\xi=0$, $z=1$ and $\xi=\pi$, $z=-1$, we should introduce other coordinates like $r = \sqrt{|\xi|^2+|z-1|^2}$, $\xi' = \xi/r$, $z' = (z-1)/r$ in the first neighbourhood.

This procedure is described in detail in Sections 8 and 9.

In the differential case the boundary matrix $C$ should satisfy [illegible]. For the boundary operator $C(E_x,\xi,z)$ we define a $n\times mn$ [illegible] boundary [illegible]

$$(5.[illegible]) \qquad \overset{\vee}{C}(\xi,z) = [C_0(\xi,z), C_1(\xi,z),\dots,C_{\nu_k}(\xi,z),0,\dots,].$$

Define also $nm\times nm$ matrices $\hat{A} = \mathrm{diag}(A,A,\dots,A)$, $\hat{B} = \mathrm{diag}(B,B,\dots,B)$. Then for the difference approximation there is a following

Condition 5.1. $\dim\overset{\vee}{C}(0,1)\mathrm{Ker}\hat{A} = \dim\overset{\vee}{C}(\pi,1)\mathrm{Ker}\hat{B} = n$.

This condition is also imposed on the matrix $B$, since for $\xi=\pi$ the operator

$$L(E_x,\pi,z) = (z-1)E_x + B^2(E_x+1)^2$$

approximates in some sense the parabolic equation $\frac{\partial u}{\partial t} = B^2\frac{\partial^2 u}{\partial x^2}$ with a singular matrix $B$.

In Section 6 we define a linearization $\overset{\vee}{L}(\kappa,\xi,z)$ of the $\kappa$-matrix $L(\kappa,\xi,z)$ and a singular eigenspace $V_\xi(\xi)$ of the singular $\kappa$-matrix $L(\kappa,\xi,1)$. It is shown also that $\dim V_\xi(\xi) = \frac{n-1}{2} + m$ for $\xi\neq 0,\pi$ [illegible], and there exists an analytic projection-valued function $\hat{P}(\xi)$ for $0\le\xi\le\pi$ acting in the $nm$-dimensional space $C^{nm}$ with $\mathrm{Ker}\,\hat{P}(\xi) = V_\xi(\xi)$ for $\xi\neq 0,\pi$ [illegible]. Our next condition is connected with [illegible].

Condition 5.2. $\dim\overset{\vee}{C}(\xi,1)\mathrm{Ker}\hat{P}(\xi) = \frac{n+1}{2}$ for any [illegible].

The last condition looks artificial but for $n=1$, [illegible] $\subset$ [illegible] + [illegible], and if $\overset{\vee}{C}(\xi,1)$ does not depend on $\xi$, Condition 5.2 [illegible] from Condition 5.1 and [illegible].

## [illegible] The Main Results.

Theorem 5.1. If [illegible] is fulfilled, then the difference approximation [illegible] is stable according to the definition [illegible] with [illegible].

Theorem 5.2. If [illegible] and conditions [illegible] are satisfied, then the difference approximation [illegible]

Theorem 5.3.

Sufficiency: If (UKC) and conditions 5.1 and 5.2 are satisfied, then for problem (5.14) estimate (5.19) holds with $|z_0| = 1$.

Necessity: If estimate (5.19) holds in a neighbourhood of $\zeta_0$ of a point $\zeta_0 = (\xi=\xi_0, z=1)$, which does not include the point $\xi=\pi$, $z=1$, then (UKC) is satisfied in this neighbourhood and condition 5.2 holds for any $\xi$ with $(\xi,1)\in\Omega(\zeta_0)$. If the point $\xi=0$, $z=1$ belongs to $\Omega(\zeta_0)$, then additionally $\dim \tilde{S}(0,1)\mathrm{Ker}\tilde{A}=1$.

In the case $\zeta_0 = (\pi,1)$ the formulation of the necessary conditions is more complicated and is given in subsection 9.3.

Let us summarize the contents of this part.

In Section 6 problem ([illegible]) is linearized and some preliminar results about the eigenvalues of the κ-matrix $L(\kappa,\xi,z)$ and about the singular eigenspace $\tilde{V}_0(\xi)$ of the linearized matrix $\tilde{L}(\kappa,\xi,z=1)$ are obtained. In Section 7 the stability problem is investigated in a neighbourhood of a point $\xi_0=0,\pi$ and $2\pi$, $z_0=1$. The κ-matrix $L(\kappa,\xi,z)$ is singular for $z=1$. So the main problem arising here is to bring the linear κ-matrix $\tilde{L}(\kappa,\xi,z)$ to a block-form near the point $z=1$. The methods applied here are similar to those used in subsection 3.3. We use very much the fact that the matrix $L(\kappa,\xi,z)$ may be represented as a product of two κ-matrices $L_1(\kappa,\xi,z)$ and $L_2(\kappa,\xi,z)$. The first matrix behaves like the λ-matrix $\lambda I+A_1+iB_1$ considered in the differential case and the second one is regular at $z=1$.

In Section 8 the neighbourhood of the point $\xi_0=0$, $z_0=1$ is considered. We introduce the inner coordinates $\zeta'=(\xi',z',r)$ where

$$r = \sqrt{|\xi|^2+|z-1|^2}, \quad \xi' = \xi/r, \quad z' = z/r.$$

Near the point $\zeta_0' = (\xi_0',0,0)$ there appear the same difficulties as in Section 7. The appropriate block structure of the matrix $L(\kappa,\xi,z)$ is studied in subsection 8.1. In subsection 8.2 we consider the problem of construction of the Kreiss symmetrizer for a block $M(\zeta')$ which is a perturbation of a simple Jordan block. Such a problem arises in a neighbourhood of a point $\zeta_0' = (\xi_0',z_0',0)$ with $\mathrm{Re}\,z_0' = 0$

and $z_0' \neq 0$. The matrix $M(\zeta')$ may be represented in a form

$$M(\zeta') = \kappa_0' I = \begin{pmatrix} e_{q-1} & 1 & 0 & .. & 0 \\ e_{q-2} & 0 & 1 & .. & 0 \\ . & & & & . \\ . & & & & 1 \\ e_0 & 0 & & .. & 0 \end{pmatrix}$$

where $e_i = e_i(\zeta')$ are analytic functions of $\zeta'$ and $\kappa_0'$ is a real constant.

It turns out that $e_i(\zeta')$ are the coefficients of a Weierstrass polynomial corresponding to one of the equations (5.26). Using only the dissipativity of the difference approximation and applying the technique of the Weierstrass preparation theorem (see [9]) we obtain some estimates about the imaginary part of $e_i(\zeta')$. Then the Kreiss construction of a symmetrizer (see [2]) may be applied directly to the matrix $M(\zeta')$.

In Section 9 we finally consider a neighbourhood of the point $\xi=\pi$, $z=1$. The inner coordinates $\zeta' = (\xi',z',r)$, where

$r = \sqrt{|\xi-\pi|^2+|z-1|}$, $\xi' = (\pi-\xi)/r$, $z' = (z-1)/r^2$, are introduced. In subsection 9.1 the block structure of the matrix $\tilde{L}(\kappa,\xi,z)$ is investigated in a neighbourhood of a point $\zeta_0' = (\xi_0',z_0',0)$ with $z_0 \neq 0$. In subsection 9.2 the singular case $z_0' = 0$ is considered. The problems arising here are similar to those studied in subsection 8.1. However now the matrix $L(\kappa,\xi,z)$ cannot be represented as a product of matrices $L_1$ and $L_2$ and therefore the situation is more complicated. In the last subsection theorems 5.1-5.3 are proved locally.

In Section 10 we discuss the results obtained in Part I and their possible generalization for other difference approximations with characteristic and non-characteristic boundary.

## 6. Preliminary Transformations and Results.

### 6.1. Linearization of the difference problem.

The difference operator $L(E_x,\zeta)$ in (5.21), which is a polynomial of order 2 in $E_x$, will be written in a form

$$L(E_x,\zeta) = \sum_{\nu=0}^{m} A_\nu(\zeta)E_x^\nu \quad , A_\nu(\zeta) = 0 \quad \text{for} \quad \nu > 2. \tag{6.1}$$

Here m is defined as in (5.5) and by $\zeta$ we denote henceforth the pair $(\xi,z)$. So, considering $L(E_x,\zeta)$ as a matrix polynomial of order m, we introduce its linearization

$$\tilde{L}(E_x,\zeta) = \tilde{A}_0(\zeta) + \tilde{A}_1(\zeta)E_x \tag{6.2}$$

where the square matrices $\tilde{A}_0$ and $\tilde{A}_1$ of order mn are defined as in (2.8). The operator $\tilde{L}(E_x,\zeta)$ acts on the space of mn-dimensional grid vector functions

(6.3) $u(x) = (u^{(1)}(x),u^{(2)}(x),\ldots,u^{(m)}(x))'$ , where $u^{(\nu)}(x)\in\mathbb{C}^n$, $\nu = 1,2,\ldots,m$.

The boundary operator $S(E_x,\zeta)$ in (5.22) (B) is replaced by a nxmn matrix

$$\tilde{S}(\zeta) = [S_0(\zeta),S_1(\zeta),\ldots,S_{\nu_1}(\zeta),0,\ldots,0] . \tag{6.4}$$

If u(x) is a solution of problem (5.22), then defining grid functions

$$\tilde{u}(x) = (u(x),E_x u(x),\ldots,E_x^{m-1}u(x)),\ \tilde{F}(x) = (0,\ldots,0,F(x)) \tag{6.5}$$

we obtain that $\tilde{u}(x)$ is a solution of the problem

(A) $\tilde{L}(E_x,\zeta)\tilde{u}(x) = \tilde{F}(x)$

(6.6)

(B) $\tilde{S}(\zeta)\tilde{u}(0) = g$

For problem (6.6) estimates (5.17)-(5.19) become correspondingly the following:

$$(6.7)\qquad \left(\frac{|z|-|z_0|}{|z|}\right)^2 \|\tilde{u}\|^2 \le K\left(\Delta t\, \frac{|z|-|z_0|}{|z|}\, |g|^2 + \|\tilde{F}\|^2\right)$$

$$(6.8)\qquad \left(\frac{|z|-|z_0|}{|z|}\right)^2 \|\tilde{u}\|^2 \le K\left(\Delta t\, \frac{|z|-|z_0|}{|z|}\, |g|^2 + \|\tilde{F}\|^2 + \frac{1}{|z|^2}\, |\tilde{F}_b|^2\right)$$

$$(6.9)\qquad \left(\frac{|z|-|z_0|}{|z|}\right)^2 \|\tilde{u}\|^2 + \Delta t\, \frac{|z|-|z_0|}{|z|}\, |R(z)\tilde{u}(0)|^2 \le K\left(\Delta t\, \frac{|z|-|z_0|}{|z|^2}\, |g|^2 + \|\tilde{F}\|^2\right).$$

*We denote here the norm in* $\ell_2(x)$ by $\|\cdot\|$ instead of $\|\cdot\|_x$ as it was in (5.17)-(5.19). The norm $|\tilde{F}_b|^2$ in (6.8) is defined as $\sum_{\nu=0}^{m} |\tilde{F}(\nu\Delta x)|^2$, and the operator R in (5.19), originally acting on the boundary values of u(x), is in a natural way applied to the mn-dimensional vector $\tilde{u}(0)$. It is easy to show that estimates (6.7)-(6.9) for problem (6.6) with arbitrary $\tilde{F}(x)$ (i.e. $\tilde{F}(x)$ not necessarily of the form (6.5)) are equivalent to the corresponding estimates (5.17)-(5.19) for problem (5.22). In the following we shall deal only with problem (6.6) and estimates (6.7)-(6.9). For simplicity of notations we remove the symbol ~ from u and F in the above problem and estimates.

## 6.2 Preliminary analysis of the κ-matrix L(κ,ζ).

To the difference operator $L(E_x,\zeta)$ in (5.21) corresponds a κ-matrix

$$(6.10)\qquad L(\kappa,\zeta) = \kappa(z-1)I + (\kappa+1)[illegible] \cdot \cos(\xi/2) \cdot [illegible] - [illegible],$$ where

$\eta = \eta(\kappa,\zeta) = A\alpha + B\beta$, $\alpha = \alpha(\kappa,\zeta) = (\kappa-1)\cos(\xi/2)$, $\beta = \beta(\kappa,\zeta) = (\kappa+1)\sin(\xi/2)$ [illegible].

According to representation (6.1) we consider L(κ,ζ) as a matrix polynomial of degree m. Then $\tilde{L}(\kappa,\zeta) = \tilde{A}_0(\zeta) + \kappa\tilde{A}_1(\zeta)$ is the linearization of $L(\kappa,\zeta)$ as described in subsection 2.2. Formula (2.10) is rewritten as

$$E(\kappa,\zeta)\tilde{L}(\kappa,\zeta)F(\kappa) = L(\kappa,\zeta)\oplus I_{(m-1)n}, \tag{6.11}$$

where the matrices $E(\kappa,\zeta)$ and $F(\kappa)$ (which should not be confused by the shift operator $E_x$ and the grid function $F(x)$) are defined as in (2.9).

In order to study the behaviour of κ-matrices $L(\kappa,\zeta)$ and $\tilde{L}(\kappa,\zeta)$ at infinity introduce

$$\tilde{L}^{(\infty)}(\kappa,\zeta) = \kappa\tilde{L}(1/\kappa,\zeta) = \kappa\tilde{A}_0(\zeta) + \tilde{A}_1(\zeta)$$

and

$$L^{(\infty)}(\kappa,\zeta) = \kappa^2 L(1/\kappa,\zeta) = \kappa(z-1)I + ((\kappa+1)/2)\cdot\cos(\xi/2)C^{(\infty)} - (C^{(\infty)})^2/2 \tag{6.12}$$

where

$$C^{(\infty)} = C^{(\infty)}(\kappa,\xi) = -A\alpha + B\beta .$$

Then similarly to (6.11) we have

$$E^{(\infty)}(\kappa,\zeta)\tilde{L}^{(\infty)}(\kappa,\zeta)F^{(\infty)}(\kappa) = I_{(m-1)n} \oplus L^{(\infty)}(\kappa,\zeta) \tag{6.13}$$

with $E^{(\infty)}(\kappa,\zeta)$ and $F^{(\infty)}(\kappa)$ defined as in (6.11).

Expression (6.10) for $L(\kappa,\zeta)$ may be considered as a polynomial $\ell(C,\kappa,\zeta)$ of second degree in C with coefficients depending on κ and ζ. So we write

$$L(\kappa,\zeta) = \ell(C,\kappa,\zeta) ,$$

and similarly

$$L^{(\infty)}(\kappa,\zeta) = \ell(C^{(\infty)},\kappa,\zeta) .$$

Consider the characteristic [illegible]

$$|\tilde{L}(\kappa,\zeta)| \quad |L(\kappa,\zeta)| = 0 . \tag{6.14}$$

According to assumption 1.1 the matrix $C = A\alpha+B\beta$ is singular for any $\alpha$ and $\beta$. Therefore, if $(z-1)\kappa = 0$, then $|L(\kappa,\zeta)| = |C/2|\cdot|(\kappa+1)\cos(\xi/2)I-C| = 0$, and for general $\kappa$ and $\zeta$ there is a factorization

$$|L(\kappa,\zeta)| = (z-1)\kappa\cdot p(\kappa,\zeta), \tag{6.15}$$

where $p(\kappa,\zeta)$ is a $\kappa$-polynomial with coefficients analytic in $\zeta$. Since $|L^{(\infty)}(0,\zeta)| = 0$, the matrix $L^{(\infty)}(\kappa,\zeta)$ has for any $\zeta$ an eigenvalue $\kappa = 0$. Hence, $L(\kappa,\zeta)$ considered as $\kappa$-matrix of degree 2 has for any $\zeta$ an eigenvalue $\kappa = \infty$. Therefore the polynomial $p(\kappa,\zeta)$ is of degree $2n-2$ at most.

Statement 6.1. The polynomial $p(\kappa,\zeta)$ is regular for any $\zeta = (\xi,z)$ with real $\xi$.

Proof. Suppose that for some $\zeta$, $p(\kappa,\zeta) \equiv 0$. By taking $\kappa = e^{i\varphi}$ one obtains

$$L(\kappa,\zeta) = \kappa[(z-1)I+2\hat{C}(i\cos(\varphi/2)\cos(\xi/2) + \hat{C})],$$

where

$$\hat{C} = A\cos(\xi/2)\sin(\varphi/2) + B\sin(\xi/2)\cos(\varphi/2).$$

The matric $\hat{C}$ is daigonalizable

$$\hat{C} \approx \operatorname{diag}(0,\lambda_1,\lambda_2,\ldots,\lambda_{n-1}),$$

where $\lambda_j$ are real, distinct and non-zero. Then

$$|L(\kappa,\zeta)| = \kappa^n(z-1)\prod_{j=1}^{n-1}[(z-1)+2\lambda_j(i\cos(\varphi/2)\cos(\xi/2)+\lambda_j)]. \tag{6.16}$$

Therefore, for some $1 \le j \le n-1$ we have

$$z-1+2\lambda_j(i\cos(\varphi/2)\cos(\xi/2) + \lambda_j) = 0 \text{ for any } \varphi.$$

If $\cos(\xi/2) \neq 0$, then taking $\varphi = \pi$ we obtain $z = 1-2\lambda_j^2$, i.e. $z$ is real. On the other hand, for $\varphi = 0$ it follows that $\operatorname{Im} z \neq 0$. If $\cos(\xi/2) = 0$, then $z = 1-2b_j^2\cos^2(\varphi/2)$ with $b_j \neq 0$, and therefore $z$ depends on $\varphi$. The above contradictions prove the statement.

For $z \neq 1$ the characteristic equation (6.14) may be written in a form

$$\kappa\, p(\kappa,\zeta) = 0 \tag{6.17}$$

and has, generally speaking, $2n-1$ finite roots including the constant root $\kappa = 0$. In order to investigate the infinite spectrum of $\tilde{L}(\kappa,\zeta)$ we consider the characteristic equation

$$|\tilde{L}^{(\infty)}(\kappa,\zeta)| = |\kappa^{m-2}\cdot L^{(\infty)}(\kappa,\zeta)| = \kappa^{(m-2)n}|L^{(\infty)}(\kappa,\zeta)| = 0 . \tag{6.18}$$

Where

$$|L^{(\infty)}(\kappa,\zeta)| = |\kappa^2 L(1/\kappa,\zeta)| = \kappa^{2n}|L(1/\kappa,\zeta)| = (z-1)\kappa(\kappa^{2n-2}p(1/\kappa,\zeta)).$$

Denoting by

$$p^{(\infty)}(\kappa,\zeta) = \kappa^{2n-2}p(1/\kappa,\zeta)$$

a $\kappa$-polynomial of degree at most $2n-2$ we rewrite equation (6.18) for $z \neq 1$ in a form

$$\kappa^{(m-2)n+1}p^{(\infty)}(\kappa,\zeta) = 0 . \tag{6.19}$$

Equation (6.19) has a constant root $\kappa = 0$ of multiplicity at least $(m-2)n+1$, and hence $\tilde{L}(\kappa,\zeta)$ has an eigenvalue $\kappa = \infty$ of the same multiplicity. So the number of all roots (counted according to their multiplicity) of equation (6.17) together with the zero root of (6.19) is equal to nm.

Statement 6.2. Let $\zeta = (\xi,z)$ with $z = 1$ and $\xi \neq 0,\pi \bmod 2\pi$, or with $|z| \geq 1$, $z \neq 1$ and any $0 \leq \xi \leq 2\pi$. Then equation (6.17) has no roots $\kappa$ with $|\kappa| = 1$.

Proof. Suppose that $\kappa = e^{i\varphi}$ is a root of equation (6.17). Then (6.16) implies that $z = 1-2\lambda_j(i\cos(\varphi/2)\cos(\xi/2)+\lambda_j)$ for some $j \geq 2$, i.e. z is an eigenvalue of the amplification matrix $G(\varphi,\xi)$. If $|z| \geq 1$, statement 5.1 implies that $z = 1$ and $\varphi = \zeta = 0,\pi \mod 2\pi$.

Statement 6.3. For $\zeta = \zeta_0 = (0,1)$ equation (6.17) has a root $\kappa = 1$ of multiplicity n-1 and, besides the simple root $\kappa = 0$, another n-1 finite roots $\kappa$ with $|\kappa| \neq 1$.

Proof. Let $\zeta = (0,z)$, $z \neq 1$. Then

$$|L(\kappa,\zeta)| = |\ell(A(\kappa-1),\kappa,\zeta)| = \kappa(z-1)\prod_{j=2}^{n}\ell(a_j(\kappa-1),\kappa,\zeta)$$

and therefore

$$p(\kappa,\zeta_0) = \prod_{j=2}^{n}\ell(a_j(\kappa-1),\kappa,\zeta_0) = \prod_{j=2}^{n}[a_j(\kappa-1)(\kappa+1-a_j(\kappa-1))/2] .$$

It is now obvious that the equation $p(\kappa,\zeta_0) = 0$ has a root $\kappa = 1$ of multiplicity n-1 and another n-1 roots of the form $\kappa_j = (a_j+1)/(a_j-1)$. Since $a_j$ are real and different from zero, $|\kappa_j| \neq 0$. Moreover, according to statement 3.1 $a_j > 0$ for $j = 2,3,\ldots,(n+1)/2$, and the rest of $a_j$ is negative. Since all $a_j$ are distinct, then $\kappa_j$ are distinct, where the first $(n-1)/2$ roots $\kappa_j$ are with $|\kappa_j| > 1$ and the remaining $\kappa_j$ have $|\kappa_j| < 1$.

Statement 6.4. For $\zeta = \zeta_1 = (\pi,1)$ equation (6.17) has a root $\kappa = -1$ of multiplicity 2n-2 and a simple root $\kappa = 0$.

Proof. As in statement 6.3 we consider $\zeta = (\pi,z)$ with $z \neq 1$. Then

$$|L(\kappa,\zeta)| = |\kappa(z-1)I + A^2(\kappa+1)^2/4| = \kappa(z-1)\prod_{j=2}^{n}(a_j^2(\kappa+1)^2/4 + \kappa(z-1)) .$$

Therefore, at the point $\zeta = \zeta_1$ equation (6.17) takes the form $\kappa(\kappa+1)^{2n-2} = 0$.

The matrix $L(\kappa,\zeta)$ may be factorized

(6.20) $$L(\kappa,\zeta)=-(1/\zeta)(s_1 I+C)(s_2 I+C)$$

where $-s_{1,2}$ are roots of the equation $\ell(s,\kappa,\zeta) = 0$. Using formula (3.3) one obtains that

$$|L(\kappa,\zeta)| = \text{const.}\ s_1 s_2 p_0(\alpha,\beta,s_1)\cdot p_0(\alpha,\beta,s_2) .$$

Since $s_1 s_2 = -2\kappa(z-1)$, we derive from (6.15)

(6.21) $$p(\kappa,\zeta) = \text{const.}\ p_0(\alpha,\beta,s_1)\cdot p_0(\alpha,\beta,s_2) .$$

If $z = 1$, then $s_1 = 0$, $s_2 = -(\kappa+1)\cos(\xi/2)$, and

(6.22) $$p(\kappa,\zeta) = \text{const.}\cdot p_0(\alpha,\beta,0)\cdot p_0(\alpha,\beta,-(\kappa+1)\cos(\xi/2)) .$$

Similarly $L^{(\infty)}(\kappa,\zeta)$ is factorized

(6.23) $$L^{(\infty)}(\kappa,\zeta) = -(1/\zeta)(s_1 I+C^{(\infty)})(s_2 I+C^{(\infty)})$$

with the same $s_{1,2}$ as in (6.20), and $p^{(\infty)}(\kappa,\zeta)$ is a product

(6.24) $$p^{(\infty)}(\kappa,\zeta) = \text{const.}\cdot p_0(-\alpha,\beta,s_1)\cdot p_0(-\alpha,\beta,s_2) .$$

Since the polynomial $p(\kappa,\zeta)$ is regular in $\kappa$ for any $\zeta = e^{i\xi}$, $z=1$, also the $\kappa$-polynomials $p_0(\alpha,\beta,0)$ and $p_0(\alpha,\beta,-(\kappa+1)\cos\xi/2)$ are regular for any $0 \le \xi \le \pi$. Using for the above polynomials the notations $p_1(\kappa,\xi)$ and $p_2(\kappa,\xi)$ one can show with the aid of (3.3) that

$$\kappa^{n-1} p_1(1/\kappa,\xi) = p_0(-\alpha,\beta,0) \qquad \text{[illegible]}$$

and

$$\kappa^{n-1} p_2(1/\kappa,\xi) = p_0(-\alpha,\beta,-(\kappa+1)\cos(\xi/2)) .$$

Consider now the κ-matrix $L(\kappa,\zeta)$ at the point $\zeta = (\xi,1)$, i.e.

$$L(\kappa,\zeta) = (C/2)\{I\cdot(\kappa+1)\cos(\xi/2)-C\} .$$

The matrix $C = C(\kappa,\xi)$ is singular of order one. The matrix $I(\kappa+1)\cos(\xi/2)-C$ is regular for $\xi \neq \pi$ and $L(\kappa,\pi,1) = K^2(\kappa+1)^2$. Therefore $L(\kappa,\zeta)$ is singular of order one for any $0 \leq \xi \leq 2\pi$. We may apply to the κ-matrix $L(\kappa,\zeta)$ the theory developed in Section 2. According to lemma 3.1 there is a vector function $\varphi_0(\alpha,\beta)$, whose components are homogeneous polynomials in α and β of some degree $q_0-1$, such that $(A\alpha+B\beta)\varphi_0(\alpha,\beta) = 0$ and $\varphi_0(\alpha,\beta)$ does not vanish for any complex α and β, $|\alpha|+|\beta| \neq 0$. Assumption 1.2 implies also that the degree $q_0-1$ is equal to $(n-1)/2$. Then $\varphi_0(\alpha,\beta)$ considered as a function of κ for a given ξ is a singular root function of $L(\kappa,\zeta)$. The degree of $\varphi_0(\alpha,\beta)$ in κ is also $q_0-1$, since the highest term of κ in $\varphi_0(\alpha,\beta)$ is $\kappa^{q_0-1}\varphi_0(\cos\xi/2, i\sin\xi/2) \neq 0$. Denote by $V_0(\xi)$ the space spanned by the vectors $\varphi_0(\alpha,\beta)$ for fixed ξ and all κ. Let $\xi \neq 0,\pi \bmod 2\pi$. Then the vector $\varphi_0(\alpha,\beta)$ is non-zero for any κ since $|\alpha|+|\beta| \neq 0$ for any κ. According to lemma 2.1, $\dim V_0(\xi) = (n+1)/2$. The vector $\varphi_0(\alpha,\beta)$ is proportional to the vector $\varphi_0(\lambda,1)$, where $\lambda = \alpha/\beta$. For fixed $\xi \neq 0,\pi \bmod 2\pi$ the function $\lambda(\kappa) = \alpha/\beta$ is a conformal mapping of the infinite complex plane on itself. Therefore the space $V_0(\xi)$ is spanned by all the vectors $\varphi(\lambda,1)$ and is independent of ξ. So for $\xi \neq 0,\pi \bmod 2\pi$ we denote $V_0(\xi)$ by $V_0$. If $\xi = 0$, $\varphi(\alpha,\beta) = \varphi(\alpha,0) \in \operatorname{Ker} A$ and $V_0(0) = \operatorname{Ker} A$. Similarly $V_0(\pi) = \operatorname{Ker} B$. Let us define

(6.26) $\tilde{\varphi}_0(\kappa,\xi) = (\varphi_0(\alpha,\beta),\kappa\varphi_0(\alpha,\beta),\ldots,\kappa^{m-1}\varphi_0(\alpha,\beta))' = F_1(\kappa)\varphi_0(\alpha,\beta)$

where, as in subsection 2.2, we denote by $F_1(\kappa)$ the first m columns of the matrix $F(\kappa)$. Then $\tilde{\varphi}_0(\kappa,\xi)$ is a singular root function of the matrix $\tilde{L}(\kappa,\xi,1)$. We also define

(6.27) $$\tilde{V}_0(\xi) = \mathrm{Sp}\{\tilde{\varphi}_0(\kappa,\xi)\}\ ,\quad \kappa \in \mathbb{C}$$

the singular eigenspace of the singular matrix $L(\kappa,\xi,1)$ of order one. The degree of $\hat{\varphi}_0(\kappa,\xi)$ in $\kappa$ is $(n-1)/2+m-1$. Since the vectors $\varphi_0(\alpha,\beta)$ and therefore $\tilde{\varphi}_0(\kappa,\xi)$ do not vanish for any $\kappa$ and $\xi \neq 0,\ \pi \bmod 2\pi$, we obtain from lemma 2.1

(6.28) $$\dim \tilde{V}_0(\xi) = (n-1)/2 + m \quad \text{for} \quad \xi \neq 0,\pi \bmod 2\pi.$$

It is obvious that $\tilde{V}_0(0) = \mathrm{Ker}\ \tilde{A}$ and $\tilde{V}_0(\pi) = \mathrm{Ker}\ \tilde{B}$, where the nmxnm matrices $\tilde{A}$ and $\tilde{B}$ are defined as in subsection 5.4.

We shall build a set of vector-functions $\psi_j(\xi)$, $j = 0,1,\ldots,(n-1)/2+m-1$, analytic and independent for $0 \le \xi \le 2\pi$, which form a basis of $\tilde{V}_0(\xi)$ for any $\xi \neq 0,\pi \bmod 2\pi$. Namely, by choosing $(n-1)/2$ distinct and not imaginary values $\lambda_1,\lambda_2,\ldots,\lambda_{(n-1)/2}$ we determine

$$\psi_j(\xi) = F_1(\kappa_j(\xi))\varphi_0(\lambda_j,1)\ ,\quad j = 1,2,\ldots,(n-1)/2$$

where

$$\kappa_j(\xi) = (\cos(\xi/2) + \lambda_j\cdot i\cdot\sin(\xi/2))/(\cos(\xi/2)-\lambda_j\cdot i\cdot\sin(\xi/2))$$

so that

$$\lambda_j = \alpha(\kappa_j(\xi),\xi)/\beta(\kappa_j(\xi),\xi).$$

Obviously $\psi_j(\xi)$ is proportional to $\tilde{\varphi}_0(\kappa_j(\xi),\xi)$ for $\xi \neq 0,\pi \bmod 2\pi$.

Let us define

[illegible] $(\xi) = \hat{\varphi}_0(0,\xi)$ and $\psi_{(n+1)/2+j}(\xi) = \frac{\partial^j}{\partial\kappa^j}\tilde{\varphi}_0(\kappa,\xi)\Big|_{\kappa=0}$, $j =$ [illegible]$,\ldots,m-1$

where

$$\tilde{\varphi}_j^{(\infty)}(\kappa,\xi) = \kappa^{(n-1)/2+m-1}\cdot\tilde{\varphi}_j(1/\kappa,\xi). \qquad ([illegible])$$

For $\xi \neq 0,\pi \bmod 2\pi$, all $\kappa_j(\xi)$, $j = 1,2,\dots,(n-1)/2$, are distinct, non-zero and finite. Then, according to corollary [illegible], the vectors $\psi_j(\xi)$, $j = 0,1,\dots,(n-1)/2+m-1$, form a basis of $\tilde{V}_0(\xi)$ for the above $\xi$. For $\xi = 0$ we have

$$\kappa_j(0) = 1,\ \psi_j(0) = F_1(1)\varphi_j(\lambda_j,1),\ j = 1,2,\dots,(n-1)/2,$$

and

$$\psi_0(0) = (\varphi_0(-1,0),0,\dots,0)'\ ,\ \psi_{(n+1)/2+j}(0) = \frac{\partial^j}{\partial\kappa^j}\left[F_m^{(\alpha)}(\kappa)\cdot\varphi_0((1-\kappa),0)\right]\Big|_{\kappa=0}$$

where $F_m^{(\infty)}(\kappa)$ is defined as in subsection [illegible]. Since $\varphi_0(\lambda,0) \in$ Ker $A$, it is easy to show that the vectors $\psi_0(0)$ and $\psi_{(n+1)/2+j}(0)$, $j = 0,1,\dots,m-2$, form a basis of the space Ker $\tilde{A}$. The vectors $\varphi_j(\lambda_j,1)$, $j = 1,2,\dots,(n-1)/2$, and $\varphi_0(-1,0)$ are independent and form a basis of the space $V_0$. Therefore all the vectors $\psi_j(0)$, $j = 0,1,\dots,(n-1)/2+m-1$ are independent and form a basis of the space $F_1(1)V_0$ + Ker $\tilde{A}$, where the sum is not direct. Similarly, one can prove that the vectors $\psi_0(\pi)$ and $\psi_{(n+1)/2+j}(\pi)$, $j = 0,1,\dots,m-2$, form a basis of the space Ker $\tilde{B}$, and all the vectors $\psi_j(\pi)$, $j = 0,1,\dots,(n-1)/2+m-1$ form a basis of the sum $F_1(-1)V_0$+Ker $\tilde{B}$. Using the basis $\{\psi_j(\xi)\}$ one can construct a projector $\tilde{P}(\xi)$ in the space $\mathbb{C}^{mn}$, which depends smoothly on $\xi$ [illegible] and satisfies

Ker $\tilde{P}(\xi) = \tilde{V}_0(\xi)$ for $\xi \neq 0,\pi$ mod $2\pi$

and

Ker $\tilde{P}(0) = F_1(1)V_0$ + Ker $\tilde{A}$, Ker $\tilde{P}(\pi) = F_1(-1)V_0$+Ker $\tilde{B}$.

## 7. The neighbourhood of a point $\zeta_0 = (\xi_0, z_0)$ with $\xi_0 \neq 0, \pi \bmod 2\pi$ and $z_0 = 1$.

If $z_0 \neq 1$ and $|z_0| \geq 1$, the κ-matrix $\tilde{L}(\kappa,\zeta_0)$ is regular and its eigenvalues κ have $|\kappa| \neq 1$, so that the investigation of stability is quite trivial. Therefore henceforth we restrict ourselves to the case $z_0 = 1$.

### 7.1. Block structure of the κ-matrix $\tilde{L}(\kappa,\zeta)$ near the point $\zeta = \zeta_0$.

Consider the characteristic equation (6.17) at the point $\zeta = \zeta_0$. Using (6.22) we can write it in a form

$$(7.1) \qquad \kappa\, p(\kappa,\zeta_0) = \kappa\, p_0(\alpha,\beta,0)\,p_0(\alpha,\beta,-\kappa+1)\cos\xi_0 \ldots = \ldots$$

where α and β correspond to κ and $\xi_0$ and are given by [illegible]. Let $\kappa_0, \kappa_1, \ldots, \kappa_r$ be all the different roots of equation (7.1) with multiplicities $q_0, q_1, \ldots, q_r$. We assume that $\kappa_0 = 1$. Let $\kappa_j$, $0 \leq j \leq r$, be a root of the polynomial $p_0(\alpha,\beta,0)$ with multiplicity $q_j^{(1)}$ and of the polynomial $p_0(\alpha,\beta,-\kappa+1)\cos$ [illegible] with multiplicity $q_j^{(2)}$. Then

$$q_j = q_j^{(1)}+q_j^{(2)} \text{ for } j \geq 1 \text{ and } q_0 = q_0^{(1)} + q_0^{(2)} + 1.$$

Similarly we consider equation (6.19) at the point $\zeta_0$

$$(7.2) \qquad \kappa^{(m-2)n+1}\, p^{(\infty)}(\kappa,\zeta_0) = \kappa^{(m-2)n+1}\, p_0(-\alpha,\beta,0)\,p_0(-\alpha,\beta,-\kappa+1) \ldots .$$

[illegible] that $\kappa_\infty = \infty$ is an infinite root of equation (7.1) with multiplicity $q_\infty$ [illegible] is a root of equation (7.2) with the same multiplicity. Similarly we define $q_\infty^{(1)}$ and $q_\infty^{(2)}$ so that

$$q_\infty = q_\infty^{(1)} + q_\infty^{(2)} + (m-2)n+1.$$

It follows from (6.25) that

$$\sum_{j=0}^{t} i_j^{(1)} + i_\infty^{(1)} = \sum_{j=0}^{t} i_j^{(2)} + i_\infty^{(2)} = n-1,$$

and therefore

$$\sum_{j=0}^{t} i_j + i_\infty = n\cdot n .$$

Consider the equation $p_0(\alpha,\beta,0) = 0$. Since $p_0(\alpha,\beta,0)$ is a homogeneous function of $\alpha$ and $\beta$ (of degree $n-1$), introducing $\lambda = (\kappa-1)/(\kappa+1)$ we can write the above equation in a form $p_0(\lambda, 1$ [illegible] $) = 0$. According to statement 3.1, the last equation has $(n-1)/2$ roots with $\mathrm{Re}\,\lambda > 0$ and the same number of roots with $\mathrm{Re}\,\lambda < 0$. Therefore the equation $p_0(\alpha,\beta,0) = 0$ has $(n-1)/2$ roots $\kappa$ with $|\kappa| < 1$ and the same number of roots (including $\kappa = \infty$ of multiplicity $i_\infty^{(1)}$) with $|\kappa| > 1$.

Let us select a small circular contour $\Gamma_j$, $j = 0,1,\ldots,t$, around the point $\kappa_j$ and denote by $\Gamma_\infty$ a contour obtained from $\Gamma_0$ by the transformation $\kappa \to 1/\kappa$. We suppose that $\Gamma_\infty$ surrounds [illegible]. Let [illegible] be a sufficiently small neighborhood of the point [illegible] in the space of the complex pairs $(\xi,z)$ such that for any $\zeta \in$ [illegible] there are no roots of equation (6.17) on the above contours $\Gamma_j$, $j = 0,1,\ldots,t,\infty$. Then for any $\zeta = (\xi,z) \in \Omega(\zeta_0)$ with $z \neq 1$ we can define mutually orthogonal projectors

$$P_j(\zeta) = (2\pi i)^{-1} \oint_{\Gamma_j} [illegible] , \quad j = 0,1,\ldots,t$$

and

$$P_\infty(\zeta) = [illegible] \oint_{\Gamma} [illegible]$$

and

[illegible]

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These projectors are obviously not defined for $z = 1$.

Let us denote by $\Omega(\kappa_j)$, $j = 0,1,\dots,t$, a small circular neighbourhood of the point $\kappa_j$ containing the contour $\Gamma_j$, and by $\Phi(\Omega(\kappa_j))$ – the space of vector functions $\varphi(\kappa)\in\mathbb{C}^{mn}$ analytic in $\Omega(\kappa_j)$. We suppose that all these neighbourhoods together with the set $\Omega(\kappa_\infty)$ obtained from $\Omega(\kappa_0)$ by the mapping $\kappa \to 1/\kappa$ are mutually disjoint. Using equivalence (6.11) we can replace the projector $P_j(\zeta)$ by an operator $Q_j(\zeta) : \Phi(\Omega(\kappa_j)) \to \mathbb{C}^{mn}$ given by

$$(7.4)\qquad Q_j(\zeta)\varphi = (2\pi i)^{-1}\oint_{\Gamma_j} F(\kappa)(L^{-1}(\kappa,\zeta)\oplus I_{(m-1)n})\varphi(\kappa)d\kappa \ .$$

Then the images of $Q_j(\zeta)$ and $P_j(\zeta)$ coincide when $z \neq 1$.

Denote $\Omega(\kappa_j,\zeta_0) = \Omega(\kappa_j)\times\Omega(\zeta_0)$. Considering the factorization in (6.20) we obtain that $s_1$ and $s_2$ are analytic functions of $\kappa$ and $\zeta$ in $\Omega(\kappa_j,\zeta_0)$, and

$$(7.5)\qquad s_1 = 2\kappa(z-1)/[(\kappa+1)\cos(\xi/2)] + O(z-1)^2 \ , \ s_2 = -(\kappa+1)\cos(\xi/2) + O(z-1).$$

Let us consider the most difficult case $q_j^{(1)} \neq 0$, $q_j^{(2)} \neq 0$. Since $|sI + C(\kappa,\xi)| \equiv s\,p_0(\alpha,\beta,s)$ and $p_0(\alpha,\beta,0) = p_0(\alpha,\beta,s_2) = 0$ for $\kappa = \kappa_j$, $\zeta = \zeta_0$, it follows that $s = 0$ and $s = -s_2(\kappa_j,\zeta_0) \neq 0$ are eigenvalues of the matrix $C(\kappa_j,\zeta_0)$ and the eigenvalue $s = 0$ is of some multiplicity $\rho > 1$. As in lemma 3.4 there is some $n\times n$ matrix $D(\kappa,\xi)$ analytic and invertible for $\kappa\in\Omega(\kappa_j)$ and $\zeta = (\xi,1)\in\Omega(\zeta_0)$, such that

$$(7.6)\qquad D^{-1}(\kappa,\xi)C(\kappa,\xi)D(\kappa,\xi) = \mathrm{diag}(N_0(\kappa,\xi),N_1(\kappa,\xi),N_2(\kappa,\xi)),$$

*where*

$$(7.7)\qquad N_0(\kappa,\xi) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & 1 \\ 0 & e_1 & e_2 & \dots & e_{\rho-1} \end{pmatrix} .$$

Here $e_k = e_k(\kappa,\xi)$, $k = 1,\dots,\rho-1$, are analytic functions of $\kappa,\xi$ with $e_k(\kappa_j,\xi_0) = 0$, the matrix $N_1(\kappa_j,\xi_0)$ has the eigenvalue $-s_2(\kappa_j,\zeta_0)$ and the eigenvalues of the matrix $N_2(\kappa_j,\xi_0)$ are different from 0 and $-s_2(\kappa_j,\zeta_0)$. We may also assume that the first column of the matrix $D(\kappa,\xi)$ is equal to $\varphi_0(\alpha,\beta)$ - the singular root function of the singular $\kappa$-matrix $C(\kappa,\xi) = A\alpha+B\beta$. It follows from (6.20) and (7.6) that

$$(7.8)\quad D^{-1}LD = -(1/2)\cdot\mathrm{diag}(s_2I+N_0, s_1I+N_1, (s_1I+N_2)(s_2I+N_2))\cdot\mathrm{diag}(s_1I+N_0, s_2I+N_1, I)$$

For the sake of brevity we omitted the arguments $\kappa$ and $\zeta$ in the above matrices. Since the first matrix on the right hand side of (7.8) is invertible, we may replace the operator $Q_j(\zeta)$ by a new one, which is denoted by the same letter

$$(7.9)\quad Q_j(\zeta)\varphi = (2\pi i)^{-1}\oint_{\Gamma_j} F(\kappa)[D(\kappa,\xi)\cdot\mathrm{diag}((s_1I+N_0)^{-1},(s_2I+N_1)^{-1},I)\oplus I_{(m-1)n}]\varphi(\kappa)d\kappa .$$

Obviously, the operator $Q_j(\zeta)$ and the projector $P_j(\zeta)$ still have the same image for $z \neq 1$. Let us define

$$(7.10)\quad Q_j^{(1)}(\zeta)\varphi = (2\pi i)^{-1}\oint_{\Gamma_j} F(\kappa)[D(\kappa,\xi)\cdot\mathrm{diag}((s_1I+N_0)^{-1},I,I)\oplus I_{(m-1)n}]\varphi(\kappa)d\kappa$$

$$(7.11)\quad Q_j^{(2)}(\zeta)\varphi = (2\pi i)^{-1}\oint_{\Gamma_j} F(\kappa)[D(\kappa,\xi)\cdot\mathrm{diag}(I,(s_2I+N_1)^{-1},I)\oplus I_{(m-1)n}]\varphi(\kappa)d\kappa$$

Lemma 7.1. a) For $z \neq 1$ the space $\mathrm{Im}\, Q_j(\zeta)$ is a direct sum of the spaces $\mathrm{Im}\, Q_j^{(1)}(\zeta)$ and $\mathrm{Im}\, Q_j^{(2)}(\zeta)$ of dimensions $q_j-q_j^{(2)}$ and $q_j^{(2)}$ respectively.
b) For $z = 1$ the space $\mathrm{Im}\, Q_j^{(2)}(\zeta)$ is a regular invariant subspace of the $\kappa$-matrix $\tilde{L}(\kappa,\zeta)$ still of the dimension $q_j^{(2)}$.

c) There is a $mn\times q_j^{(2)}$ matrix valued function $X_j^{(2)}(\zeta)$ analytic in $\Omega(\zeta_0)$, whose columns form a basis of $\mathrm{Im}\,Q_j^{(2)}(\zeta)$ for any $\zeta\in\Omega(\zeta_0)$, and there is also a $q_j^{(2)}\times q_j^{(2)}$ matrix-valued function $M_j^{(2)}(\zeta)$ analytic in $\Omega(\zeta_0)$ such that

$$(7.12)\qquad \tilde{A}_1(\zeta)X_j^{(2)}(\zeta)M_j^{(2)}(\zeta)+\tilde{A}_0(\zeta)X_j^{(2)}(\zeta)=0\ ,$$

where $M_j^{(2)}(\zeta_0)$ is a Jordan matrix with the eigenvalue $\kappa_j$.

Proof: It follows from (7.9)-(7.11) that

$$(7.13)\qquad \mathrm{Im}\,Q_j^{(1)}(\zeta)\subset \mathrm{Im}\,Q_j(\zeta)\quad\text{and}\quad \mathrm{Im}\,Q_j^{(2)}(\zeta)\subset \mathrm{Im}\,Q_j(\zeta)\ .$$

On the other hand the operators $Q_j(\zeta)$, $Q_j^{(1)}(\zeta)$ and $Q_j^{(2)}(\zeta)$ are unchanged if the unit matrices (except $s_1I$ and $s_2I$) in formulas (7.9)-(7.11) are replaced by zero. But then obviously

$$(7.14)\qquad Q_j(\zeta)=Q_j^{(1)}(\zeta)+Q_j^{(2)}(\zeta)$$

and also

$$(7.15)\quad Q_j^{(1)}(\zeta)(\mathrm{Ker}\,Q_j^{(2)}(\zeta))=\mathrm{Im}\,Q_j^{(1)}(\zeta)\ ,\ Q_j^{(2)}(\zeta)(\mathrm{Ker}\,Q_j^{(1)}(\zeta))=\mathrm{Im}\,Q_j^{(2)}(\zeta)\ .$$

It follows from (7.13) and (7.14) that

$$(7.16)\qquad \mathrm{Im}\,Q_j(\zeta)=\mathrm{Im}\,Q_j^{(1)}(\zeta)+\mathrm{Im}\,Q_j^{(2)}(\zeta).$$

In order to prove that the above sum is direct, one should show that the equality $Q_j^{(1)}(\zeta)\varphi_1+Q_j^{(2)}(\zeta)\varphi_2=0$ implies $Q_j^{(1)}(\zeta)\varphi_1=Q_j^{(2)}(\zeta)\varphi_2=0$. By (7.15) we may replace $\varphi_1$ and $\varphi_2$ by some $\varphi\in\mathfrak{O}(\Omega(\kappa_j))$ such that $Q_j^{(1)}(\zeta)\varphi=Q_j^{(1)}(\zeta)\varphi_1$ and

$Q_j^{(2)}(\zeta)\varphi = Q_j^{(2)}(\zeta)\varphi_2$, and hence $Q_j(\zeta)\varphi = 0$. Let us denote the whole nmxnm matrices under the integral sign in (7.9)-(7.11) by $L_0^{-1}(\kappa,\zeta)$, $L_1^{-1}(\kappa,\zeta)$ and $L_2^{-1}(\kappa,\zeta)$ respectively. Then the matrices $L_0(\kappa,\zeta)$ and $L_1(\kappa,\zeta)$ for $z \neq 1$, and the matrix $L_2(\kappa,\zeta)$ for all z are right divisors of the matrix $\tilde{L}(\kappa,\zeta)$. Since the integral $\oint_{\Gamma_j} L_0^{-1}(\kappa,\zeta)\varphi(\kappa)d\kappa$ is zero and $\tilde{L}(\kappa,\zeta)L_0^{-1}(\kappa,\zeta)\varphi(\kappa)$ is analytic, it follows by lemma 2.4 that $L_0^{-1}(\kappa,\zeta)\varphi(\kappa)$ is analytic in $\Omega(\kappa_j)$. But then also $L_1^{-1}(\kappa,\zeta)\varphi(\kappa)$ and $L_2^{-1}(\kappa,\zeta)\varphi(\kappa)$ are analytic in $\Omega(\kappa_j)$, and therefore $Q_j^{(1)}(\zeta)\varphi = Q_j^{(2)}(\zeta)\varphi = 0$. The $\kappa$-matrix $L_1(\kappa,\zeta)$ for $\zeta\in\Omega(\zeta_0)$ with $z \neq 1$, and the $\kappa$-matrix $L_2(\kappa,\zeta)$ for any $\zeta\in\Omega(\zeta_0)$ have correspondingly $q_j-q_j^{(2)}$ and $q_j^{(2)}$ eigenvalues surrounded by the contour $\Gamma_j$. Therefore for $z \neq 1$, according to remark 2.5, the dimensions of Im $Q_j^{(1)}(\zeta)$ and of Im $Q_j^{(2)}(\zeta)$ are correspondingly $q_j-q_j^{(2)}$ and $q_j^{(2)}$. (Note that for $j \neq 0$, $q_j-q_j^{(2)} = q_j^{(1)}$ but $q_0-q_0^{(2)} = q_0^{(1)}+1$).

In order to prove part (b) of the lemma one should show that for $z = 1$ the matrix $L_2(\kappa,\zeta)$ satisfies the conditions of lemma 2.5. It remains only to verify that any eigenvector $\varphi$ of $L_2(\kappa,\zeta)$ corresponding to an eigenvalue $\kappa\in\Omega(\kappa_j)$ is not a singular eigenvector of $\tilde{L}(\kappa,\zeta)$. But the vector $\varphi$ may be written in a form $\varphi = F(\kappa)(D(\kappa,\xi)\oplus 1_{(m-1)n})\psi$, where the components of the vector $\psi$ corresponding to the block $N_0$ are zero. On the other hand the singular eigenvector of $\tilde{L}(\kappa,\zeta)$ is given by

$$\tilde{\varphi}(\kappa,\xi) = F(\kappa)(D(\kappa,\xi)\oplus 1_{(m-1)n})\cdot(1,0,\ldots,0)' = F_1(\kappa)\varphi_0(\alpha,\beta) .$$

Therefore the above vector $\varphi$ is not proportional to $\tilde{\varphi}_0(\kappa,\xi)$.

Since the operator $Q_j^{(2)}(\zeta)$ is analytic in $\Omega(\zeta_0)$ and Im $Q_j^{(2)}(\zeta)$ is of the constant dimension $q_j^{(2)}$, there is some basis in Im $Q_j^{(2)}(\zeta)$, which depends analytically on $\zeta\in\Omega(\zeta_0)$. So we define the matrix $X_j^{(2)}(\zeta)$ such that its columns form the above basis. We may assume that $X_j^{(2)}(\zeta) = Q_j^{(2)}(\zeta)\Psi(\kappa)$, where $\Psi(\kappa)$ is a $nmxq_j^{(2)}$ matrix whose columns belong to $\Phi(\Omega(\kappa_j))$. Then

$$\tilde{A}_0(\zeta)X_j^{(2)}(\zeta) = -\tilde{A}_1(\zeta)Q_j^{(2)}(\zeta)(\kappa\Psi(\kappa)) .$$

The matrix $Q_j^{(2)}(\zeta)(\kappa\Psi(\kappa))$ may be represented as $X_j^{(2)}(\zeta)M_j^{(2)}(\zeta)$, where the matrix $M_j^{(2)}(\zeta)$ is analytic in $\Omega(\zeta_0)$. The matrix $L_2(\kappa,\zeta_0)$ has in $\Omega(\kappa_j)$ only the eigenvalue $\kappa = \kappa_j$ of multiplicity $q_j^{(2)}$. According to lemma 2.5 we may assume that the columns of $X_j^{(2)}(\zeta_0)$ form a regular Jordan sequence of $\tilde{L}(\kappa,\zeta_0)$ corresponding to the eigenvalue $\kappa = \kappa_j$. In this case the matrix $M_j(\zeta_0)$ is a Jordan matrix with the eigenvalue $\kappa_j$. The lemma is completely proved.

The operator $Q_j^{(1)}(\zeta)$, unlike $Q_j^{(2)}(\zeta)$, is not analytic in $\Omega(\zeta_0)$. However in the analogy to lemma 3.4 we can prove the following

Lemma 7.2. a) The space Im $Q_j^{(1)}(\zeta)$ depends analytically on $r\in\Omega(\zeta_0)$, i.e. there exists an $mnxq_j^{(1)}$ matrix valued function $X_j^{(1)}(\zeta)$ analytic in $\Omega(\zeta_0)$, whose columns form a basis of Im $Q_j^{(1)}(\zeta)$ for $z \neq 1$ and are independent also for $z = 1$.

b) For $\zeta = (\xi,1)\in\Omega(\zeta_0)$ the columns of $X_j^{(1)}(\zeta)$ belong to the singular eigenspace $\tilde{V}_0(\xi)$ and for $\zeta = \zeta_0$ they form a singular Jordan chain of length $q_j^{(1)}$ generated by the singular root function $\tilde{\varphi}_0(\kappa,\xi_0)$ at the point $\kappa = \kappa_j$.

c) There is a $q_j^{(1)} \times q_j^{(1)}$ matrix valued function $M_j^{(1)}(\zeta)$ analytic in $\Omega(\zeta_0)$ such that

$$\tilde{A}_1(\zeta)X_j^{(1)}(\zeta)M_j^{(1)}(\zeta) + \tilde{A}_0(\zeta)X_j^{(1)}(\zeta) = 0 \tag{7.17}$$

and the matrix $M_j^{(1)}(\zeta_0)$ is a Jordan cell with the eigenvalue $\kappa_j$.

If $j = 0$, the number $q_j^{(1)}$ in the statement and in the proof of this lemma should be replaced by $q_j^{(1)}+1$.

Proof: The operator $Q_j^{(1)}(\zeta)$ in (7.10) may be written in a form

$$Q_j^{(1)}(\zeta)\varphi = (2\pi i)^{-1} \oint_{\Gamma_j} F_1(\kappa)D(\kappa,\xi)[(s_1 I+N_0(\kappa,\xi))^{-1} \oplus 0_{n-\rho}]\varphi(\kappa)d\kappa \tag{7.18}$$

where the vector $\varphi(\kappa)$ is now n-dimensional. If we multiply the whole integrand in (7.18) on the left by $\tilde{L}(\kappa,\zeta)$, we still get an analytic function in $\Omega(\kappa_j,\zeta_0)$. Note that the first column of the matrix $F_1(\kappa)D(\kappa,\xi)$ is the singular root function $\tilde{\varphi}_0(\kappa,\xi)$. Compairing the determinants $|s_1 I+C(\kappa,\xi)| = s_1 p_0(\alpha,\beta,s_1)$ and $|s_1 I+N_0(\kappa,\xi)| = s_1(\pm e_1 \pm e_2 s_1^1 \pm \ldots \pm e_{\ell-1} s_1^{\rho-1})$ for $(\kappa,\xi) \in \Omega(\kappa_j,\zeta_0)$, one obtains that the equation $p_0(\alpha,\beta,0) = 0$ at the point $\zeta = \zeta_0$ is equivalent to the equation $e_1(\kappa,\zeta_0) = 0$. Therefore, as in lemma 3.4, we obtain that $e_1(\kappa,\zeta_0) = f_1(\kappa)(\kappa-\kappa_j)^{q_j^{(1)}}$, where $f_1(\kappa_j) \neq 0$. Let us multiply the matrix $s_1 I + N_0(\kappa,\xi)$ on the left by matrices $E_1$, $E_2$, $E_3$ and $E_4$, where $E_2 = \mathrm{diag}(1/s_1,1,1,\ldots,1)$ for $j \neq 0$ and $E_2 = \mathrm{diag}(\kappa/s_1,1,1,\ldots,1)$ for $j = 0$ and the rest of the matrices are defined as in lemma 3.4. We arrive at a matrix $N_0'(\kappa,\zeta)$, which is analytic in $\Omega(\kappa_j,\zeta_0)$, the inverse matrix $(N_0'(\kappa,\zeta))^{-1}$ is analytic for $(\kappa,\zeta) \in \Gamma_j \times \Omega(\zeta_0)$ and for $z = 1$ we have

$$(N_0'(\kappa,\zeta))^{-1} = \operatorname{diag}(f(\kappa)/e_1(\kappa,\xi),1,1,\dots,1) \text{ for } j \neq 0$$

(7.19)

$$(N_0'(\kappa,\zeta))^{-1} = \operatorname{diag}(f(\kappa)/(\kappa e_1(\kappa,\xi)),1,1,\dots,1) \text{ for } j = 0.$$

Therefore

$$(N_0'(\kappa,\zeta_0))^{-1} = \operatorname{diag}((\kappa-\kappa_j)^{-q_j^{(1)}},1,1,\dots,1). \tag{7.20}$$

Let us replace the operator $Q_j^{(1)}(\zeta)$ in (7.18) by a new one which is again denoted by $Q_j^{(1)}(\zeta)$ :

$$Q_j^{(1)}(\zeta)\varphi = (2\pi i)^{-1} \oint_{\Gamma_j} F_1(\kappa)D(\kappa,\xi)[(N_0'(\kappa,\zeta)^{-1} \oplus 0_{n-\rho}]\varphi(\kappa)d\kappa . \tag{7.21}$$

The above operator $Q_j^{(1)}(\zeta)$ depends analytically on $\zeta$ in $\Omega(\zeta_0)$. Since the matrices $\Psi_k(\kappa,\zeta)$, $k = 1,2,3,4$, are invertible for $z \neq 1$, the operators in (7.18) and in (7.21) have for $z \neq 1$ the same images of the dimension

$$\dim(\operatorname{Im} Q_j^{(1)}(\zeta)) = q_j^{(1)}. \tag{7.22}$$

It is clear that the integrand in (7.21) multiplied on the left by $\tilde{\Psi}(\kappa,\zeta)$ become analytic in $\Omega(\kappa_j,\zeta_0)$. It follows from (7.19) that for $\zeta = (\xi,1)$ the image of $Q_j^{(1)}(\zeta)$ is spanned by the vectors $\tilde{\varphi}_0(\kappa,\zeta)$ with different $\kappa$ and, therefore, belongs to the space $\tilde{V}_0(\xi)$. As in lemma 3.4 we determine n-dimensional vector functions

$$\psi_k(\kappa) = ((\kappa-\kappa_j)^{q_j^{(1)}-k-1},0,0,\dots,0)' \text{ for } k = 0,1,\dots,q_j^{(1)}-1$$

and a matrix $\Psi(\kappa)$ built from the columns $\psi_k(\kappa)$. Then the matrix $X_j^{(1)}(\zeta)$ is defined by

$$X_j^{(1)}(\zeta) = Q_j(\zeta)(\Psi(\kappa)) . \tag{7.23}$$

Since $Q_j(\zeta_0)\psi_k(\kappa) = \frac{1}{k!}\frac{\partial^k\tilde{\varphi}_0(\kappa,\xi_0)}{\partial\kappa^k}\Big|_{\kappa=\kappa_j}$ , the columns of $X_j^{(1)}(\zeta_0)$ form a singular Jordan chain of length $q_j$ generated by the root function $\tilde{\varphi}_0(\kappa,\xi_0)$ at the point $\kappa = \kappa_j$ . As already noted in the beginning of this subsection, the equation $p_0(\alpha,\beta,0)$ has $(n-1)/2$ roots $\kappa$ with $|\kappa| < 1$ and the same number of roots with $|\kappa|>1$. Therefore $q_j^{(1)} \leq (n-1)/2$ and $q_0^{(1)} \leq (n-1)/2+1$. Formula (6.28) implies that $q_j^{(1)} \leq \dim \tilde{V}_0(\zeta)$ even for $j = 0$, and according to lemma 2.1 the columns of $X_j^{(1)}(\zeta_0)$ are independent. The neighbourhood $\Omega(\zeta_0)$ may be chosen so small that the columns of $X_j^{(1)}(\zeta)$ are independent for any $\zeta\in\Omega(\zeta_0)$ and, according to (7.22), form a basis of the space Im $Q_j^{(1)}(\zeta)$ for $z \neq 1$. The last statement of this lemma is proved as in lemma 7.1.

The case $j = \infty$ requires a separate consideration. Due to the equivalence in (6.13) formula (7.4) is replaced by

$$Q_\infty(\zeta)\varphi = (2\pi i)^{-1} \oint_{\Gamma_0} F^{(\infty)}(\kappa)[I_{(m-1)n}\oplus(\kappa^{m-2}L^{(\infty)}(\kappa,\zeta))^{-1}]\varphi(\kappa)d\kappa \; . \tag{7.24}$$

Transformation (7.6) is replaced by a similar transformation for the matrix $C^{(\infty)}(\kappa,\zeta)$. Using (6.23) we change the definition of operators in (7.9)-(7.11) to

$$\begin{aligned}Q_\infty(\zeta)\varphi = (2\pi i)^{-1} \oint_{\Gamma_0} & F^{(\infty)}(\kappa)[I_{(m-1)n}\oplus D(\kappa,\xi) \\ & \cdot\operatorname{diag}((s_1I+N_0)^{-1},(s_2I+N_1)^{-1},I)\kappa^{2-m}]\varphi(\kappa)d\kappa\end{aligned} \tag{7.25}$$

$$\begin{aligned}Q_\infty^{(1)}(\zeta)\varphi = (2\pi i)^{-1} \oint_{\Gamma_0} & F^{(\infty)}(\kappa)[I_{(m-1)n}\oplus D(\kappa,\xi) \\ & \cdot\operatorname{diag}((s_1I+N_0)^{-1}\kappa^{2-m},I,I)]\varphi(\kappa)d\kappa\end{aligned} \tag{7.26}$$

$$(7.27)\quad Q_\infty^{(2)}(\zeta)\varphi = (2\pi i)^{-1}\oint_{\Gamma_0} F^{(\infty)}(\kappa)[I_{(m-1)n}\oplus(D(\kappa,\xi)$$
$$\cdot\mathrm{diag}(I,(s_2I+N_1)^{-1}\kappa^{2-m},I\kappa^{2-m}))]\varphi(\kappa)d\kappa\ .$$

Lemma 7.1 is still valid with the only difference that

$$(7.28)\qquad \dim(\mathrm{Im}\ Q_\infty^{(1)}(\zeta)) = q_\infty^{(1)} + 1 + (m-2)\rho \ \text{ for } z \neq 1 \text{ and}$$

$$(7.29)\qquad \dim(\mathrm{Im}\ Q_\infty^{(2)}(\zeta)) = q_\infty^{(2)} + (m-2)(n-\rho) \quad \text{for any} \quad \zeta\in\Omega(\zeta_0)\ .$$

The presence of the factor $\kappa^{2-m}$ in (7.26) makes the investigation of $Q_\infty^{(1)}(\zeta)$ more complicated. Applying to the matrix $s_1I + N_0(\kappa,\xi)$ the same transformations as in lemma 7.2 for the case $j = 0$, we replace the operator $Q_\infty^{(1)}(\zeta)$ by a new one, which is again denoted by $Q_\infty^{(1)}(\zeta)$:

$$(7.30)\qquad Q_\infty^{(1)}(\zeta)\varphi = (2\pi i)^{-1}\oint_{\Gamma_0} F_m^{(\infty)}(\kappa)D(\kappa,\xi)[\kappa^{2-m}(N_0'(\kappa,\zeta))^{-1}\oplus 0_{n-\rho}]\varphi(\kappa)d\kappa.$$

The above operator depends analytically on $\zeta\in\Omega(\zeta_0)$ and has the same image as the operator in (7.26) so that formula (7.28) is still valid. We shall prove the following

Lemma 7.3. a) There exists an $mn\times(q_\infty^{(1)}+1+(m-2)\rho)$ matrix valued function

$$(7.31)\qquad \tilde{X}_\infty^{(1)}(\zeta) = (\tilde{X}_\infty^{(1,1)}(\zeta),\tilde{X}^{(1,2)}(\zeta))$$

analytic in $\Omega(\zeta_0)$, whose columns form a basis of Im $Q_\infty^{(1)}(\zeta)$ for any $\zeta\in\Omega(\zeta_0)$.

b) For $\zeta = (\xi,1)\in\Omega(\zeta_0)$ the columns of $\tilde{X}_\infty^{(1,1)}(\zeta)$ belong to the singular eigenspace $\tilde{V}_0(\xi)$ and for $\zeta = \zeta_0$ they form a singular Jordan chain of length $q_\infty^{(1)}+m-1$ generated by the singular root function $\tilde{\varphi}_0^{(\infty)}(\kappa,\xi_0)$ at the point $\kappa = 0$.

The columns of $\tilde{X}_{\infty}^{(1,2)}(\zeta_0)$ form a regular Jordan sequence of $\kappa$-matrix $\tilde{L}^{(\infty)}(\kappa,\zeta_0)$ corresponding to the eigenvalue $\kappa = 0$ of this matrix.

c) There is a matrix-valued function $\tilde{M}_{\infty}^{(1)}(\zeta)$ analytic in $\Omega(\zeta_0)$ such that

$$\tilde{A}_1(\zeta)\tilde{X}_{\infty}^{(1)}(\zeta)\tilde{M}_{\infty}^{(1)}(\zeta) + \tilde{A}_0(\zeta)\tilde{X}_{\infty}^{(1)}(\zeta) = 0 \tag{7.32}$$

and the matrix $\tilde{M}_{\infty}^{(1)}(\zeta_0)$ has the only eigenvalue $\kappa = 0$. According to partition (7.31) the matrix $\tilde{M}_{\infty}^{(1)}(\zeta)$ may be written in a form

$$\tilde{M}_{\infty}^{(1)}(\zeta) = \begin{pmatrix} \tilde{M}_{\infty}^{(1,1)}(\zeta) & \tilde{M}_{\infty}^{(1,2)}(\zeta) \\ \tilde{M}_{\infty}^{(2,1)}(\zeta) & \tilde{M}_{\infty}^{(2,2)}(\zeta) \end{pmatrix} \text{, where} \tag{7.33}$$

$$\tilde{M}_{\infty}^{(2,1)}(\zeta) = \tilde{M}_{\infty}^{(1,2)}(\zeta) = 0 \text{ for } z = 1 . \tag{7.34}$$

Proof: Let us determine a sequence of n-dimensional vector functions

$$\psi_{1,k}(\kappa) = (\kappa^{q_{\infty}^{(1)}+m-2-k},0,0,\ldots,0)' \text{ , } k = 0,1,\ldots,q_{\infty}^{(1)}+m-2$$

and $\rho-1$ sequences

$$\psi_{2,k}(\kappa) = (0,\kappa^{m+3-k},0,\ldots,0)',\ldots,\psi_{\rho,k}(\kappa) = (\underbrace{0,0,\ldots,\kappa}_{\rho}{}^{m-3-k},0,\ldots,0)',k=0,1,\ldots,m-3.$$

The sequence of columns $\{\psi_{1,k}(\kappa)\}$ form a matrix $\Psi_1(\kappa)$ and the sequences $\{\psi_{2,k}(\kappa)\},\ldots,\{\psi_{\rho,k}(\kappa)\}$ form an $n$x$(\rho-1)$ matrix $\Psi_2(\kappa)$. We define

$$\tilde{X}_\infty^{(1,1)}(\zeta) = Q_\infty^{(1)}(\zeta)(\Psi_1(\kappa)) \ , \ \tilde{X}_\infty^{(1,2)}(\zeta) = Q_\infty^{(1)}(\zeta)(\Psi_2(\kappa))$$

and

$$\tilde{X}_\infty^{(1)}(\zeta) = (\tilde{X}_\infty^{(1,1)}(\zeta), \ \tilde{X}_\infty^{(1,2)}(\zeta)).$$

Let us note that the first column of the matrix $F_m^{(\infty)}(\kappa)D(\kappa,\xi)$ is the singular root function $\tilde{\varphi}_0^{(\infty)}(\kappa,\xi))$. We denote the next $\rho-1$ columns of this matrix by $\tilde{\varphi}_2(\kappa,\xi),\tilde{\varphi}_3(\kappa,\xi),\ldots,\tilde{\varphi}_\rho(\kappa,\xi)$. Obviously, the last columns are regular root functions of $\tilde{L}^{(\infty)}(\kappa,\xi,1)$ of multiplicity m-2 corresponding to $\kappa = 0$, and the eigenvectors $\tilde{\varphi}_2(0,\xi),\tilde{\varphi}_3(0.\xi),\ldots,\tilde{\varphi}_\rho(0,\xi)$ are independent of the singular eigenvector $\tilde{\varphi}_0^{(\infty)}(0,\xi)$. Using (7.20), where $q_j^{(1)}$ should be replaced by $q_\infty^{(1)}+1$ and $\kappa_j$ by 0, one obtains that the columns of $\tilde{X}_\infty^{(1,1)}(\zeta_0)$ form a singular Jordan chain as proposed in part b) of the lemma, and the columns of $\tilde{X}_\infty^{(1,2)}(\zeta_0)$ form a regular Jordan sequence of $\tilde{L}^{(\infty)}(\kappa,\zeta_0)$ corresponding to $\kappa = 0$. Since $q_\infty^{(1)} \leq (n-1)/2$, it follows that $q_\infty^{(1)} + m-1 < (n-1)/2 + m = \dim V_0(\xi)$. According to lemma 2.1 the columns of $\tilde{X}_\infty^{(1,1)}(\zeta_0)$ are independent, and lemma 2.2 implies that also the columns of $\tilde{X}_\infty^{(1)}(\zeta_0)$ are independent. We shall choose the neighbourhood $\Omega(\zeta_0)$ small enough so that the columns of $\tilde{X}_\infty^{(1)}(\zeta)$ are independent for any $\zeta\in\Omega(\zeta_0)$. It follows then from (7.28) that the columns of $\tilde{X}_\infty^{(1)}(\zeta)$ form a basis of the space Im $Q_\infty^{(1)}(\zeta)$ ($Q_\infty^{(1)}(\zeta)$ is defined by (7.30)) for any $\zeta\in\Omega(\zeta_0)$). The matrix $\tilde{M}_\infty^{(1)}(\zeta)$ is obtained as in lemma 7.1. The diagonal form (7.19) of the matrix $(N_0'(\kappa,\zeta))^{-1}$ for $z = 1$ implies that the columns of $\tilde{X}_\infty^{(1)}(\xi,1)$ belong to the space $\tilde{V}_0(\xi)$ and the matrices $\tilde{M}_\infty^{(2,1)}$ and $\tilde{M}_\infty^{(1,2)}$ satisfy (7.34).

Let us return to the operator $Q_\infty^{(2)}(\zeta)$ in (7.27). Instead of the notations

$X_j^{(2)}(\zeta)$ and $M_j^{(2)}(\zeta)$ in lemma 7.1 we use for $j = \infty$ the notations $\tilde{X}_\infty^{(2)}(\zeta)$ and $\tilde{M}_\infty^{(2)}(\zeta)$. Denote the whole mnxmn integrand matrix in (7.27) by $L_2^{-1}(\kappa,\zeta)$. Then the matrix $L_2(\kappa,\zeta_0)$ has in $\Omega(\kappa_0)$ the only eigenvalue $\kappa = 0$. The corresponding eigenvectors are linear combinations of some of the last $n-\rho$ columns $\tilde{\varphi}_{\rho+1}(0,\xi_0),\dots,\tilde{\varphi}_n(0,\xi_0)$ of the matrix $\tilde{F}_m(0)D(0,\xi_0)$. According to lemma 2.5 we may assume that the columns of $\tilde{X}_\infty^{(2)}(\zeta_0)$ form a Jordan sequence of $L_2(\kappa,\zeta_0)$ corresponding to the eigenvalue $\kappa = 0$. We have shown also in lemma 7.3 that the columns of the matrix $\tilde{X}_\infty^{(1,2)}(\zeta_0)$ form a regular Jordan sequence of $\tilde{L}^{(\infty)}(\kappa,\zeta_0)$ corresponding to $\kappa = 0$ with eigenvectors $\tilde{\varphi}_2(0,\xi_0),\dots,\tilde{\varphi}_\rho(0,\xi_0)$. Since the vectors $\{\tilde{\varphi}_\kappa(0,\xi_0)\}_{k=2}^n$ are independent of $\tilde{\varphi}_0^{(\infty)}(0,\xi_0)$, the columns of $(\tilde{X}_\infty^{(1,2)}(\zeta_0), \tilde{X}_\infty^{(2)}(\zeta_0))$ form a regular Jordan sequence of $\tilde{L}^{(\infty)}(\kappa,\zeta_0)$ corresponding to $\kappa = 0$. Let us denote

$$X_\infty^{(1)}(\zeta) = \tilde{X}_\infty^{(1,1)}(\zeta)\ ,\ X_\infty^{(2)}(\zeta) = (\tilde{X}_\infty^{(1,2)}(\zeta),\ \tilde{X}^{(2)}(\zeta)),\ X_\infty(\zeta) = (X_\infty^{(1)}(\zeta),X_\infty^{(2)}(\zeta))$$

$$(7.35)\quad M_\infty^{(1,1)}(\zeta) = \tilde{M}_\infty^{(1,1)}(\zeta),\ M_\infty^{(1,2)}(\zeta) = (\tilde{M}_\infty^{(1,2)}(\zeta),0),\ M_\infty^{(2,1)}(\zeta) = (0,\tilde{M}_\infty^{(2,1)}(\zeta))'$$

$$M_\infty^{(2,2)}(\zeta) = \begin{pmatrix} \tilde{M}_\infty^{(2,2)}(\zeta) & 0 \\ 0 & \tilde{M}_\infty^{(2)}(\zeta) \end{pmatrix}\ ,\quad M_\infty(\zeta) = \begin{pmatrix} M_\infty^{(1,1)}(\zeta)\ , & M_\infty^{(1,2)}(\zeta) \\ M_\infty^{(2,1)}(\zeta) & M_\infty^{(2,2)}(\zeta) \end{pmatrix}.$$

We can summarize the above results for the case $j = \infty$ in the following

Lemma 7.4.a) The columns of the matrix $X_\infty(\zeta)$ are analytic vector functions in $\Omega(\zeta_0)$ and form for $z \neq 1$ a basis of the space Im $P_\infty(\zeta)$.

b) For $\zeta = (\xi,1)\in\Omega(\zeta_0)$ the columns of $X_\infty^{(1)}(\zeta)$ belong to the singular eigenspace $\tilde{V}_0(\xi)$ and for $\zeta = \zeta_0$ they form a singular Jordan chain of length $q_\infty^{(1)}+m-1$

generated by the singular root function $\tilde{\varphi}_0^{(\infty)}(\kappa,\xi_0)$ at the point $\kappa = 0$.

c) The columns of $X_\infty^{(2)}(\zeta_0)$ form a regular Jordan sequence of $\tilde{L}^{(\infty)}(\kappa,\zeta_0)$ corresponding to $\kappa = 0$.

d) The matrix $M_\infty(\zeta)$ in (7.35) is analytic in $\Omega(\zeta_0)$ with $M_\infty^{(2,1)}(\xi,1) = M_\infty^{(1,2)}(\xi,1) = 0$ and satisfies the identity

(7.36) $$\tilde{A}_0(\zeta)X_\infty(\zeta) + \tilde{A}_1(\zeta)X_\infty(\zeta)M_\infty(\zeta) = 0 .$$

Let us define the following matrix valued functions (we omit the variable $\zeta$)

(7.37) $X_j = (X_j^{(1)},X_j^{(2)})$, $j = 0,1,\dots,t$; $X_{F1} = (X_1,X_2,\dots,X_t)$, $X_F = (X_0,X_{F1})$,

$$X = (X_F,X_\infty) .$$

According to the partition of the matrices $X_j$, the matrices $X_{F1},X_F$ and $X$ are also partitioned as

(7.38) $$X_{F1} = (X_{F1}^{(1)}, X_{F1}^{(2)}),\ X_F = (X_F^{(1)}, X_F^{(2)}),\ X = (X^{(1)},X^{(2)}) .$$

In the same way we define matrices

$$M_j = M_j^{(1)}\oplus M_j^{(2)},\ M_{F1} = \mathrm{diag}(M_1,M_2,\dots,M_t),\ M_F = M_0\oplus M_{F1}$$

and the partitions

(7.39) $$M_{F1} = M_{F1}^{(1)}\oplus M_{F1}^{(2)},\ M_F = M_F^{(1)}\oplus M_F^{(2)} .$$

We shall denote the matrices $M_\infty^{(1,1)}$ and $M_\infty^{(2,2)}$ also by $M_\infty^{(1)}$ and $M_\infty^{(2)}$ respectively. The finite eigenvalues $\kappa_j$, $j = 0,1,\dots,t$, split up into two groups: the group I which contains $\kappa_j$ with $|\kappa_j| < 1$ and the group II with $|\kappa_j| > 1$. Then the matrix

$X_I$ consists of all the matrices $X_j$ with $\kappa_j \in I$ and the matrix $X_{II}$ is defined analogously. The corresponding partial blocks $M_I$ and $M_{II}$ of $M_F$ are determined in a natural way. We suppose that the matrices $X_I$, $X_{II}$ and $M_I$, $M_{II}$ are also partitioned

$$X_I = (X_I^{(1)}, X_I^{(2)}), \quad M_I = (M_I^{(1)} \oplus M_I^{(2)}) \tag{7.40}$$

and similarly for $X_{II}$ and $M_{II}$.

Let us denote

$$T = (\tilde{A}_1 X_F, \; \tilde{A}_0 X_\infty) \; . \tag{7.41}$$

We shall partition the rows of the inverse matrix $T^{-1}$ according to the columns of X and use the similar notations. For example:
matrix $X_I^{(1)}$ corresponds to the matrix $(T^{-1})_I^{(1)}$.

Now we introduce the mnxmn dimensional matrices

$$\tilde{B}_0 = (-M_F) \oplus I \quad \text{and} \quad \tilde{B}_1 = I \oplus (-M_\infty) \; . \tag{7.42}$$

Using (7.12), (7.17) and (7.36) one obtains the following main identity

$$\tilde{L}(\kappa,\zeta) X(\zeta) = T(\zeta)(\tilde{B}_0(\zeta) + \kappa \tilde{B}_1(\zeta)). \tag{7.43}$$

Lemma 7.5. a) The matrix $X(\zeta)$ is non singular for $z \neq 1$.
b) For $\zeta = (\xi,1) \in \Omega(\zeta_0)$ the columns of the matrix $(X_{II}^{(1)}(\zeta), X_\infty^{(1)}(\zeta))$ together with the first column of $X_0^{(1)}(\zeta)$ form a basis of the space $\tilde{V}_0(\xi)$. Similarly, the columns of the matrix $X_I^{(1)}(\zeta)$ together with the first m-1 columns of $X_\infty^{(1)}(\zeta)$ form a basis of $\tilde{V}_0(\xi)$.

c) For $\zeta = (\xi,1) \in \Omega(\zeta_0)$ the columns of $X^{(2)}(\zeta)$ are independent of the space $\tilde{V}_0(\xi)$ and therefore independent of the columns of $X^{(1)}(\zeta)$. Hence, the columns of the matrix $X_I(\zeta) = (X_I^{(1)}(\zeta), X_I^{(2)}(\zeta))$ are independent.

Proof. Since the columns of $X_j(\zeta)$, $j = 0,1,\dots,t,\infty$, form for $z \neq 1$ a basis of the space $\operatorname{Im} P_j(\zeta)$, the first statement of the lemma follows from the spectral theory of regular linear $\lambda$-matrices. As shown in the beginning of this subsection

$$\sum_I q_j^{(1)} = \sum_{II} q_j^{(1)} + q_\infty^{(1)} = \frac{n-1}{2} \quad ,$$

where the sums $\sum_I$ and $\sum_{II}$ are taken over $j = 0,1,\dots,t$, with $\kappa_j$ belonging respectively to the groups I and II. Then the matrix $(X_{II}^{(1)}(\zeta), X_\infty^{(1)}(\zeta))$ has $\sum_{II} q_j^{(1)} + q_\infty^{(1)} + m-1 = (n-1)/2 + m-1$ columns. Adding to these columns the first column of $X_0^{(1)}$ we obtain sequence of $(n-1)/2+m$ vectors, which consists at the point $\zeta = \zeta_0$ of singular Jordan chains generated by the singular root functions $\tilde{\varphi}_0(\kappa,\xi_0)$. Then, according to (6.28) and corollary 2.1, these vectors form a basis of the space $\tilde{V}_0(\xi_0)$. If the neighbourhood $\Omega(\zeta_0)$ is small enough, the statement in the last sentence remains true when $\zeta_0$ is replaced by any $\zeta = (\xi,1) \in \Omega(\zeta_0)$. In the same way one proves the second statement of b). According to lemmas 7.1 and 7.4 the columns of $X^{(2)}(\zeta_0)$ form a regular Jordan sequence of the $\kappa$-matrix $\tilde{L}(\kappa,\zeta_0)$. By lemma 2.2 these columns are independent of the singular space $\tilde{V}_0(\zeta_0)$. Again, if $\Omega(\zeta_0)$ is small enough, the last statement is true for any $\zeta = (\xi,1) \in \Omega(\zeta_0)$.

Denote $\hat{T}^{-1}(\zeta) = (z-1)T^{-1}(\zeta)$. The rows of the matrix $\hat{T}^{-1}(\zeta)$ are partitioned according to the matrix $T^{-1}(\zeta)$. Analogously to lemma 3.6 we have the following

Lemma 7.6 a) The matrix valued functions $\hat{T}^{-1}(\zeta)$ and $(T^{-1}(\zeta))^{(2))}$ are analytic in $\Omega(\zeta_0)$.

b) The last row of the matrix $(\hat{T}^{-1}(\zeta_0))_j^{(1)}$, $j = 0,1,\dots,t,\infty$, is non-zero.

Proof. The analyticity of $\hat{T}^{-1}(\zeta)$ follows as in lemma 3.6 from stability of the Cauchy problem. We should merely replace the functions $\varphi_j(\lambda')$ by

$$\varphi_j(\kappa) = |\kappa I-M_F(\zeta)|/|\kappa I-M_j(\zeta)| \quad \text{for} \quad |\kappa_j| < 1$$

$$\varphi_j(\kappa) = \kappa^{-1}|I-\kappa^{-1}M_F(\zeta)|\cdot|M_\infty(\zeta)-\kappa^{-1}I|/|I-\kappa^{-1}M_j(\zeta)| \quad \text{for} \quad |\kappa_j| > 1$$

and

$$\varphi_\infty(\kappa) = \kappa^{-1}|I-\kappa^{-1}M_F(\zeta)|$$

and integrate $X(\zeta)(\tilde{B}_0(\zeta)+\kappa\tilde{B}_1(\zeta))^{-1}\varphi_j(\kappa)T^{-1}(\zeta)$ around the unit circle $|\kappa| = 1$.

Let $\zeta= (\xi,1)\in\Omega(\zeta_0)$. As in lemma 3.6 we have

$$\operatorname{Im} \hat{T}^{-1}(\zeta) = \operatorname{Ker} T(\zeta) .$$

Let us fix some $\kappa$ different from the eigenvalues of the $\kappa$-matrix $\tilde{B}_0(\zeta)+\kappa\tilde{B}_1(\zeta)$ for any $\zeta\in\Omega(\zeta_0)$. Then we obtain from (7.43)

$$T(\zeta) = \tilde{L}(\kappa,\zeta)X(\zeta)(\tilde{B}_0(\zeta)+\kappa\tilde{B}_1(\zeta))^{-1} .$$

Let $v\in\operatorname{Ker}T(\zeta)$ and $u = (\tilde{B}_0(\zeta)+ \tilde{B}_1(\zeta))^{-1}v$. We suppose that the components of the vectors u and v are partitioned according to the columns of $X(\zeta)$. The kernel of $\tilde{L}(\kappa,\zeta)$ consists of vectors $\tilde{\varphi} = F_1(\kappa)\varphi$ , where $\varphi\in\operatorname{Ker}L(\kappa,\zeta) = \operatorname{Ker}(C\cdot(s_2I+C))$. Since the matrix $s_2I+C$ is invertible at the point $(\kappa,\zeta)$ and the kernel of $C(\kappa,\xi)$ is spanned by $\varphi_0(\alpha,\beta)$, we obtain that the above vector $\tilde{\varphi}$ is proportional to $\tilde{\varphi}_0(\kappa,\xi)\in\tilde{V}_0(\xi)$. Therefore $X(\zeta)u\in\tilde{V}_0(\zeta)$. Since the columns of $X^{(1)}(\zeta)$ belong to $\tilde{V}_0(\xi)$ and those of $X^{(2)}(\zeta)$ are independent of $\tilde{V}_0(\xi)$, we conclude that $u^{(2)} = 0$. But

$$v^{(2)} = \begin{pmatrix} \kappa I-M_F^{(2)}(\zeta) & 0 \\ 0 & -\kappa M_\infty^{(2)}(\zeta)+I \end{pmatrix} u^{(2)} = 0$$

and hence $(\hat{T}^{-1}(\zeta))^{(2)} = 0$. Therefore $(T^{-1}(\zeta))^{(2)}$ is analytic in $\Omega(\zeta_0)$.

Part b) of the lemma is proved as in lemma 3.6.

## 7.2. Proof of theorems 5.1-5.3 in the neighbourhood $\Omega(\zeta_0)$.

Let us consider problem (6.6) in the neighbourhood $\Omega(\zeta_0)$ of the point $\zeta_0 = (\xi_0, 1)$, where $\xi_0 \neq 0, \pi \mod 2\pi$. We remove the symbol ∿ from u and F and write

$$
(7.44)\quad
\begin{array}{ll}
\text{(A)} & \tilde{L}(E_x, \zeta)u(x) = F(x), \; x = \nu\Delta x, \; \nu = 0,1,\ldots\,. \\
\text{(B)} & \tilde{S}(\zeta)u(0) = g
\end{array}
$$

Denote $v(x) = X^{-1}(\zeta)u(x)$, $G(x) = T^{-1}(\zeta)F(x)$. We suppose that the components of the vectors $v(x)$ and $G(x)$ are partitioned according to the columns of $X(\zeta)$ and use the natural notations for the partial vectors. Problem (7.44) may now be written as

$$
\text{(A)}\quad (E_x - M_F(\zeta))v_F(x) = G_F(x)
$$

$$
(7.45)\quad \text{(B)}\quad (I - M_\infty(\zeta)E_x)v_\infty(x) = G_\infty(x)
$$

$$
\text{(C)}\quad \tilde{S}(\zeta)X_I(\zeta)v_I(0) + \tilde{S}(\zeta)X_{II}(\zeta)v_{II}(0) + \tilde{S}(\zeta)X_\infty(\zeta)v_\infty(0) = g
$$

Define symmetrizers

$$
R_F(\zeta) = \begin{pmatrix} R_I(\zeta) & 0 \\ 0 & R_{II}(\zeta) \end{pmatrix} \quad \text{with } R_I(\zeta) = -cI, \; R_{II}(\zeta) = I,
$$

where c is a positive constant, and

$$
R_\infty(\zeta) = I.
$$

We may assume that

$$
M_I^*(\zeta)M_I(\zeta) \leq (1-\delta)I, \; M_\infty^*(\zeta)M_\infty(\zeta) \leq (1-\delta)I \;\text{ and }\; M_{II}^*(\zeta)M_{II}(\zeta) \geq (1+\delta)I.^{*)}
$$

---

*) See footnote in subsection 3.2.

Then the symmetrizers $R_F(\zeta)$ and $R_\infty(\zeta)$ satisfy

$$M_F^*(\zeta)R_F(\zeta)M_F(\zeta)-R_F(\zeta) \geq \delta I \quad \text{and} \quad R_\infty(\zeta)-M_\infty^*(\zeta)R_\infty(\zeta)M_\infty(\zeta) \geq \delta I. \tag{7.46}$$

Here, as usual, we denote by $\delta$ different positive constants.

Let us apply to equation (7.45) (A) the generalized energy method with the symmetrizer $R_F$. Namely, multiplying equation (7.45) (A) on the left by $R_F(-E_x-M_F)v_F$ in the sense of the scalar product in $\ell_2(x)$ and taking real part one obtains

$$<(M_F^*R_FM_F-R_F)v_F,v_F> + v_F^*(0)R_Fv_F(0)\Delta x = -\mathrm{Re}<R_F(E_x+M_F)v_F,G_F>.$$

Now it follows easily that

$$\delta\cdot\|v_F\|^2 + (|v_{II}(0)|^2-c|v_I(0)|^2)\Delta x \leq K\cdot\|G_F\|^2. \tag{7.47}$$

Similarly, multiplying equation (7.45) (B) on the left by $R_\infty(I+M_\infty E_x)v_\infty$ and taking real part we have

$$<(R_\infty-M_\infty^*R_\infty M_\infty)E_xv_\infty,E_xv_\infty> + v_\infty^*(0)R_\infty v_\infty(0)\Delta x = \mathrm{Re}<R_\infty(I+M_\infty E_x)v_\infty,G_\infty>$$

and therefore

$$\delta\|E_xv_\infty\|^2 + |v_\infty(0)|^2\Delta x \leq K\|G_\infty\|^2. \tag{7.48}$$

Adding (7.47) and (7.48) and using that $\|v_\infty\|^2 = \|E_xv_\infty\|^2 + |v_\infty(0)|^2\Delta x$ we arrive at

$$\delta\|v\|^2 + (|v_{II}(0)|^2 + |v_\infty(0)|^2-c|v_I(0)|^2)\Delta x \leq K\|G\|^2. \tag{7.49}$$

Lemma 7.7. The condition (UKC) in the neighbourhood $\Omega(\zeta_0)$ is equivalent to the condition $\det \mathfrak{S}(\zeta_0)X_I(\zeta_0) \neq 0$.

Proof: The general solution of equation (7.44) (A) for $F = 0$ is given by

$$\varphi(x_\nu,\zeta) = (\varphi_1(x_\nu,\zeta),\varphi_2(x_\nu,\zeta),\dots,\varphi_n(x_\nu,\zeta))v_I(0) = X_I(\zeta)M_I^\nu(\zeta)v_I(0)\ .$$

The nm-dimensional column vectors $\varphi_j(0,\zeta)$, $j = 1,2,\dots,n$, form the matrix $X_I(\zeta)$ and are independent and analytic in $\Omega(\zeta_0)$. Therefore the matrix $N(\xi,z)$ in (5.30) may be identified with $\tilde{S}(\zeta)X_I(\zeta)$. So if $\det N(\xi,z) \geq \delta$ for any $\zeta\in\Omega(\zeta_0)$ with $|z| > 1$, then $\det \tilde{S}(\zeta_0)X_I(\zeta_0) \neq 0$. Choosing $\Omega(\zeta_0)$ small enough, we obtain that the converse in also true.

Now we are able to prove estimate (6.7) with $|z_0| = 1$. Let (UKC) be satisfied, i.e. $\det \tilde{S}(\zeta_0)X_I(\zeta_0) \neq 0$. Then it follows from (7.45) (C) that

$$|v_I(0)| \leq K(|v_{II}(0)| + |v_\infty(0)| + |g|)\ . \tag{7.50}$$

By setting the constant c in (7.49) small enough one obtains

$$\|v\|^2 \leq K(\|G\|^2 + |g|^2\Delta x)\ .$$

Since $\|G\|^2 = \|T^{-1}(\zeta)F\|^2 \leq \dfrac{K\|F\|^2}{|z-1|^2}$ and $\|u\|^2 = \|X(\zeta)v\|^2 \leq K\|v\|^2$ we derive an estimate

$$\|u\|^2 \leq K\left(\frac{\|F\|^2}{|z-1|^2} + |g|^2\Delta x\right) \tag{7.51}$$

which is obviously stronger than (6.7) for $|z_0| = 1$.

Let us now prove in $\Omega(\zeta_0)$ the sufficiency part of theorem 5.3. We define the operator P in (6.9) for $\zeta\in\Omega(\zeta_0)$ as equal to the projector $\tilde{P}(\xi)$ in (6.30). Introduce grid vector functions

$$\hat{v} = (\hat{v}^{(1)},\hat{v}^{(2)})' = ((z-1)v^{(1)},v^{(2)})' \text{ and } \hat{G} = (\hat{G}^{(1)},\hat{G}^{(2)})' = ((z-1)G^{(1)},G^{(2)})'.$$

Then equations (7.45) (A), (B) may be written as

$$(7.52)\qquad \begin{aligned} &(A)\quad (E_x - M_F(\zeta))\hat{v}_F(x) = \hat{G}_F(x) \\ &(B)\quad (I-\hat{M}_\infty(\zeta)E_x)\hat{v}_\infty(x) = \hat{G}_\infty(x) \end{aligned}$$

where

$$\hat{M}_\infty(\zeta) = \begin{pmatrix} M_\infty^{(1)}(\zeta) & M_\infty^{(1,2)}(\zeta)\cdot(z-1) \\ M_\infty^{(2,1)}(\zeta)/(z-1) & M_\infty^{(2,2)}(\zeta) \end{pmatrix} .$$

According to part d) of lemma 7.4 the matrix $M^{(2,1)}(\zeta)/(z-1)$ is analytic in $\Omega(\zeta_0)$. The matrix $\hat{M}_\infty(\zeta)$ has the same eigenvalues as $M_\infty(\zeta)$. Hence, there exists a summetrizer $\hat{R}_\infty(\zeta)$ such that

$$\hat{R}_\infty(\zeta) - \hat{M}_\infty^*(\zeta)\hat{R}_\infty(\zeta)\hat{M}_\infty(\zeta) \geq \delta I \quad \text{and} \quad \hat{R}_\infty(\zeta) \geq I .$$

Applying to equations (7.52) the generalized energy method with the symmetrizers $R_F(\zeta)$ and $\hat{R}_\infty(\zeta)$, we get in analogy to (7.49) an estimate

$$(7.53)\qquad \delta\|\hat{v}\|^2 + (|\hat{v}_{II}(0)|^2 + |\hat{v}_\infty(0)|^2 - c|\hat{v}_I(0)|^2)\Delta x \leq K(\|\hat{G}\|^2) .$$

According to condition 5.2, $\dim \tilde{\hat{S}}(\xi,1)\tilde{\hat{V}}_0(\xi) = (n+1)/2$. Since the $(n+1)/2$ columns of the matrix $\tilde{\hat{S}}(\xi,1)X_I^{(1)}(\xi,1)$ are independent, they form a basis of the space $\tilde{\hat{S}}(\xi,1)\tilde{\hat{V}}_0(\xi)$. Then using (UKC) we obtain as in (7.50) an estimate

$$(7.54)\qquad |\hat{v}_I(0)| \leq K(|\hat{v}_{II}(0)|^2 + |\hat{v}_\infty(0)|^2 + |g|^2) .$$

Choosing the constant c in (7.53) small enough we arrive at

$$|\hat{v}(0)|^2\Delta x \leq K(\|\hat{G}\|^2 + |g|^2\Delta x) .$$

But

$$\|\hat{G}^{(1)}\|^2 = \|(T^{-1}(\zeta))^{(1)}F\|^2 \le K\|F\|^2$$

and

$$\|\hat{G}^{(2)}\|^2 = \|(T^{-1}(\zeta))^{(2)}F\|^2 \le K\|F\|^2$$

so that

$$|\hat{v}(0)|^2\Delta x \le K(\|F\|^2 + |g|^2\Delta x) .$$

Since

$$|\tilde{P}(\xi)u(0)|^2 = |\tilde{P}(\xi)X^{(1)}(\zeta)v^{(1)}(0) + \tilde{P}(\xi)X^{(2)}(\zeta)v^{(2)}(0)|^2 \le K|\hat{v}(0)|^2,$$

we have

$$\frac{|\tilde{P}(\xi)u(0)|^2\Delta x}{|z|-1} \le K\left(\frac{\|F\|^2}{|z|-1} + \frac{|g|^2\Delta x}{|z|-1}\right) .$$

Adding the last inequality to (7.51) and replacing $\frac{1}{|z-1|}$ by $\frac{1}{|z|-1}$ we arrive finally at estimate (6.9).

Consider problem (7.44) with $F = 0$ and let the estimate

$$\|u\|^2 \le K\frac{|z|}{|z|-|z_0|}|g|^2\Delta x \tag{7.55}$$

hold for any $|z| \ge |z_0| = 1+\alpha_0\Delta x$ with $\alpha_0>0$. Obviously, estimate (7.55) is weaker than estimate (6.8). We shall show that (UKC) is then satisfied in [illegible]. Indeed, if $\tilde{P}(\zeta_0)X_I(\zeta_0)v_I(0) = 0$ and $v_I(0) \ne 0$, we define for [illegible] $= (\xi_0, z)\in\Omega(\zeta_0)$ a homogeneous solution of equations (7.45) (A), (B) as

$$u(x_\nu) = u(x_\nu,z) = X_I(\xi_0,z)M_I^\nu(\xi_0,z)v_I(0).$$

[illegible]

$$\|u(x,z)\|^2 \ge K|v_I(0)|^2\Delta x \quad \text{and} \quad |\tilde{P}(\xi_0,z)u(0,z)|^2 \le K|z-1|^2 .$$

[illegible] we know from [illegible] that

$$|v_I(0)|^2 \le \frac{K|z-1|}{|z-z_0|} \quad \text{for positive } z \text{ with } |z| \ge |z_0| .$$

Taking $z = 1 + 2\alpha_0 \Delta x$ and $\Delta x$ tending to zero, we obtain that $v_I(0) = 0$.

To accomplish the proof of theorem 5.3 one should show that if estimate (6.9) is fulfilled, then dim $\tilde{S}(\zeta_0)\tilde{V}_0(\xi_0) = (n+1)/2$. Consider problem (7.44) with $g = 0$. It follows from (7.52) that

$$\hat{v}_{II}(0) = -\sum_{\nu=0}^{\infty} M_{II}^{-\nu-1}(\zeta)\hat{G}_{II}(x_\nu) \quad \text{and} \quad \hat{v}_\infty(0) = \sum_{\nu=0}^{\infty} \hat{M}_\infty^{\nu}(\zeta)\hat{G}_\infty(x_\nu) \ .$$

For fixed $F \in \ell_2(x)$ we consider $\hat{v}_{II}(0)$ and $\hat{v}_\infty(0)$ as functions of $\zeta \in \Omega(\zeta_0)$ and denote them by $\hat{v}_{II}(0,\zeta)$ and $\hat{v}_\infty(0,\zeta)$. Since the matrix $\tilde{S}(\zeta_0)X_I(\zeta_0)$ is invertible, the function $\hat{v}_I(0,\zeta)$ may be computed with the aid of the boundary condition (7.45) (C). Then $\hat{v}_I^{(1)}(0,\zeta)$ and $(z-1)\hat{v}_I^{(2)}(0,\zeta)$ depend analytically on $\zeta \in \Omega(\zeta_0)$. The vector $(\hat{v}_{II}^{(1)}(0,\zeta_0), \hat{v}_\infty^{(1)}(0,\zeta_0))'$ is given by

$$(\hat{v}_{II}^{(1)}(0,\zeta_0), \hat{v}_\infty^{(1)}(0,\zeta_0))' = Q(F) = \sum_{\nu=0}^{\infty} \begin{bmatrix} (M_{II}^{(1)}(\zeta_0))^{-1} & 0 \\ 0 & M_\infty^{(1)}(\zeta_0) \end{bmatrix}^{\nu} \cdot \begin{bmatrix} -M_{II}^{(1)}(\zeta_0)(\hat{T}^{-1}(\zeta_0))_{II}^{(1)} \\ (\hat{T}^{-1}(\zeta_0))_\infty^{(1)} \end{bmatrix} F(x_\nu) \tag{7.56}$$

We consider $Q$ as a linear operator acting on the space $\ell_2(x)$ of grid vector functions with values in $\mathbb{C}^{m-1+(n-1)/2}$. Analogously to lemma 3.7 we have the following

Lemma 7.8. The operator $Q$ is an epimorphism.

Proof: The space Im $Q$ is obviously an invariant space of the matrix $(M_{II}^{(1)}(\zeta_0))^{-1} \oplus M_\infty^{(1)}(\zeta_0)$ containing the image of the operator

$$[-M_{II}^{(1)}(\zeta_0)(\hat{T}^{-1}(\zeta_0))_{II}^{(1)} \ , \quad (\hat{T}^{-1}(\zeta_0))_{\infty}^{(1)}]' \ .$$

Since the matrix $(M_{II}^{(1)}(\zeta_0))^{-1}$ consists of matrices $(M_j^{(1)}(\zeta_0))^{-1}$ with different eigenvalues $\kappa_j^{-1}$, the space ImQ is a direct sum of invariant subspaces of matrices $(M_j^{(1)}(\zeta_0))^{-1}$, $M_{\infty}^{(1)}(\zeta_0)$ containing respectively the images of $M_j^{(1)}(\zeta_0)(\hat{T}^{-1}(\zeta_0))_j^{(1)}$ and of $(\hat{T}^{-1}(\zeta_0))_{\infty}^{(1)}$. According to part b) of lemma 7.6 the last row of the matrices $M_j^{(1)}(\zeta_0)(\hat{T}^{-1}(\zeta_0))_j^{(1)}$ and $(\hat{T}^{-1}(\zeta_0))_{\infty}^{(1)}$ is non-zero. Recalling that $M_j^{(1)}(\zeta_0)$ is a Jordan cell, one can prove now the lemma without difficulties.

Let us write the boundary condition (7.45) (C) in the form

$$\tilde{S}(\zeta)X^{(1)}(\zeta)\hat{v}^{(1)}(0,\zeta) + \tilde{S}(\zeta)X^{(2)}(\zeta)\hat{v}^{(2)}(0,\zeta)\cdot(z-1) = 0 \ .$$

Suppose that $\tilde{S}(\zeta_0)X^{(1)}(\zeta_0)\hat{v}^{(1)}(0,\zeta_0) \neq 0$. Then $\hat{v}^{(2)}(0,\zeta) = O^*((z-1)^{-1})$ (we denote $f = O^*(g)$ if $0 < \delta \leq |f/g| \leq K$). Since

$$\tilde{P}(\xi)u(0) = \tilde{P}(\xi)(X^{(1)}(\zeta)v^{(1)}(0,\zeta)+X^{(2)}(\zeta)v^{(2)}(0,\zeta))$$

$$= O(\hat{v}^{(1)}(0,\zeta)) + O^*(v^{(2)}(0,\zeta)) = O^*((z-1)^{-1}),$$

taking z-1 positive we arrive at a contradiction with the estimate

$$|\tilde{P}(\xi)u(0)|^2\Delta x < K \frac{|z|}{|z|-1} \|F\|^2 .$$

Therefore $\tilde{S}(\zeta_0)X^{(1)}(\zeta_0)\hat{v}^{(1)}(0,\zeta_0) = 0$. For suitable $F \in \ell_2(x)$ according to lemma 7.8 we may obtain any value of the vector $(\hat{v}_{II}^{(1)}(0,\zeta_0),\hat{v}_{\infty}^{(1)}(0,\zeta_0))'$. Since the columns of $X^{(1)}(\zeta_0)$ span the space $\tilde{V}_0(\xi_0)$, the space $\tilde{S}(\zeta_0)\tilde{V}_0(\xi_0)$ is spanned by $(n+1)/2$ independent columns of the matrix $\tilde{S}(\zeta_0)X_I^{(1)}(\zeta_0)$. Thus, theorem 5.3 is proved locally in $\Omega(\zeta_0)$.

## 8. The neighbourhood of the point $\zeta_0 = (0,1)$.

Let us introduce the notations

$$(8.1)\quad r = \sqrt{|\xi|^2+|z-1|^2}\ ,\ \xi' = \xi/r,\ z' = (z-1)/r,\ \zeta' = (\xi',z',r),\ \kappa' = (\kappa-1)/r\ .$$

By $\zeta_0' = (\xi_0',z_0',0)$ we denote a point with real coordinate $\xi_0'$ and complex $z_0'$ satisfying $\mathrm{Re}z_0' \geqslant 0$ and $|\zeta_0'| = 1$. Then $\Omega(\zeta_0')$ denotes a neighbourhood of $\zeta_0'$ in the three dimensional complex space $\mathbb{C}^3$ of points $\zeta' = (\xi',z',r)$, and $\Omega_R(\zeta_0')$ consists of points $\zeta'\in\Omega(\zeta_0')$ with real $\xi'$, positive $r$ and complex $z'$ such that $|z| = |1+rz'| > 1$. By $\Omega(\zeta_0)$ we denote a neighbourhood of the point $\zeta_0 = (0,1)$ in the two-dimensional complex space of pairs $\zeta = (\xi,z)$ and $\Omega_R(\zeta_0)$ consists of points $\zeta\in\Omega(\zeta_0)$ with real $\xi$ and $|z|>1$. To any point $\zeta' = (\xi',z',r)\in\mathbb{C}^3$ and any complex $\kappa'$ correspond $\zeta = (\xi,z)$ and $\kappa$ given by

$$(8.2)\qquad \xi = \xi'\cdot r,\ z = 1+rz',\ \kappa = 1+r\kappa'\ .$$

We consider problem (6.6) locally in a neighbourhood $\Omega(\zeta_0')$. Then one can select a finite number of such neighbourhoods, which cover some neighbourhood $\Omega_R(\zeta_0)$.

### 8.1 Block structure of the $\kappa$-matrix $\tilde{L}(\kappa,\zeta)$ in a neighbourhood $\Omega(\zeta_0')$.

According to statement 6.3 the characteristic equation $\kappa p(\kappa,\zeta_0) = 0$ has a root $\kappa = 1$ of multiplicity $n-1$, a simple root $\kappa = 0$, and $n-1$ different roots $\kappa_j = (a_j+1/(a_j-1)$ for $j = 2,3,\dots,n$. Equation (6.19) has for any $\zeta\in\Omega(\zeta_0)$ a root $\kappa = 0$ of multiplicity $(m-2)n+1$. Therefore $\kappa_\infty = \infty$ is an eigenvalue of $\tilde{L}(\kappa,\zeta)$ of the above multiplicity. In order to describe the roots $\kappa$ near $\kappa = 1$ as $\zeta$ tends to $\zeta_0$ we introduce $\kappa'$-matrices

$$C'(\kappa',\xi') = C(\kappa,\xi)/r = A\alpha'+B\beta',\ \text{where}$$

$$(8.3)\qquad \alpha' = \alpha'(\kappa',\xi') = \kappa'\cos(\xi/2),\ \beta' = \beta'(\kappa',\xi') = i(\kappa+1)\sin(\xi'r/2)/r,$$

and

$$L'(\kappa',\zeta') = L(\kappa,\zeta)/r = z'\kappa + C'(\kappa',\xi')\left(\frac{\kappa+1}{2}\cos(\xi/2)-\frac{r}{2}C'(\kappa',\xi')\right) .$$

The values of $\zeta$ and $\kappa$ in (8.3) are given by (8.2). Obviously $L'(\kappa',\zeta')$ is a matrix polynomial in $\kappa'$ of degree 2 depending analytically on the parameter $\zeta'\in\Omega(\zeta_0')$. For $r = 0$ we have

$$C'(\kappa',\xi') = A\kappa' + Bi\xi' \text{ and } L'(\kappa',\zeta') = z' + C'(\kappa',\xi') . \tag{8.4}$$

Using factorization (6.20) one obtains

$$L'(\kappa',\zeta') = -(1/2)(s_1'I+C')(s_2I+C) \tag{8.5}$$

where $s_1' = s_1/r$ depends analytically on $\kappa'$ and $\zeta'$, and $s_2$ depends analytically on $\kappa$ and $\zeta\in\Omega(\zeta_0)$. From (7.5) we get

$$s_1' = z'\left[\frac{2\kappa}{(\kappa+1)\cos(\xi/2)} + 0(z-1)\right]$$

and for $r = 0$, $s_1' = z'$ and $s_2 = -2$.

The characteristic equation $|L'(\kappa',\zeta')| = 0$ in neighbourhood of the point $\kappa = 1$, $\zeta = \zeta_0$ is equivalent to the equation $|s_1'I+C'| = s_1'p_0(\alpha',\beta',s_1') = 0$ which in turn is equivalent for $z' \neq 0$ to the equation

$$p_0(\alpha',\beta',s_1') = 0 . \tag{8.6}$$

For $\zeta' = \zeta_0'$ the above equation has a form

$$p_0(\kappa',i\xi_0',z_0') = 0 \tag{8.7}$$

and is regular according to $\kappa'$ also for $z_0' = 0$.

The last equation was investigated in subsection 3.1. If $z_0' = 0$ or $\mathrm{Re}z_0' \neq 0$, it has $(n-1)/2$ roots with $\mathrm{Re}\,\kappa' > 0$ and the same number of roots with $\mathrm{Re}\,\kappa' < 0$. Therefore imaginary roots $\kappa'$ are possible in equation (8.7) only for $\mathrm{Re}z_0' = 0$, $z_0' \neq 0$. It is worthwhile to note here that if $\xi_0' = 0$, the roots $\kappa'$ are non-zero since $z_0' \neq 0$. Let $\kappa_1', \kappa_2', \ldots, \kappa_t'$ be the different roots of equation (8.7) of multiplicities $q_1^{(1)}, q_2^{(1)}, \ldots, q_t^{(1)}$. We select small neighbourhoods $\Omega(\zeta_0')$ and $\Omega(\zeta_0)$ such that for any $\zeta' \in \Omega(\zeta_0')$ the corresponding point $\zeta$ belongs to $\Omega(\zeta_0)$. Denote by $\Omega(\kappa_j')$ a small neighbourhood of a point $\kappa_j'$, $j=1,2,\ldots,t$, and by $\Omega(\kappa_k)$ a small neighbourhood of a point $\kappa_k$, $k = 0,2,3,4,\ldots,n$. In the neighbourhoods $\Omega(\kappa_j')$ and $\Omega(\kappa_k)$ we select correspondingly circular contours $\Gamma_j'$ around $\kappa_j'$ and $\Gamma_k$ around $\kappa_k$. Then $\Omega_0(\kappa_j')$ and $\Omega_0(\kappa_k)$ are neighbourhoods bounded by $\Gamma_j'$ and $\Gamma_k$ respectively. The neighbourhoods $\Omega(\zeta_0')$ and $\Omega(\zeta_0)$ are supposed to be small enough so that any root $\kappa$ of equation (8.6) belongs for $\zeta' \in \Omega(\zeta_0')$ to some $\Omega_0(\kappa_j')$ and the remaining $n-1$ eigenvalues $\kappa$ of $L(\kappa,\zeta)$ belong for $\zeta \in \Omega(\zeta_0)$ to the neighbourhoods $\Omega_0(\kappa_k)$. For $z \neq 1$, $\zeta' \in \Omega(\zeta_0')$, we define as in (7.3) mutually orthogonal projectors

$$P_j^{(1)}(\zeta') = (2\pi i)^{-1} \oint_{\kappa' \in \Gamma_j'} \tilde{L}^{-1}(\kappa,\zeta)\tilde{A}_1(\zeta)d\kappa \ , \quad j = 1,2,\ldots,t$$

$$P_k(\zeta) = (2\pi i)^{-1} \oint_{\kappa \in \Gamma_k} \tilde{L}^{-1}(\kappa,\zeta)\tilde{A}_1(\zeta)d\kappa \ , \quad k = 0,2,3,\ldots,n \tag{8.8}$$

$$P_\infty(\zeta) = (2\pi i)^{-1} \oint_{\kappa \in \Gamma_0} (\tilde{L}^{(\infty)}(\kappa,\zeta))^{-1}\tilde{A}_0(\zeta)d\kappa$$

so that

$$\sum P_j^{(1)}(\zeta') + \sum P_k(\zeta) + P_\infty(\zeta) = I \ .$$

The projectors $P_j^{(1)}(\zeta')$ may be written in a form

$$
\begin{aligned}
P_j^{(1)}(\zeta') &= (2\pi i)^{-1} \oint_{\kappa' \in \Gamma_j'} F(\kappa)[L^{-1}(\kappa,\zeta) \oplus I_{(m-1)n}]E(\kappa,\zeta)\tilde{A}_1(\zeta) r d\kappa' \\
&= (2\pi i)^{-1} \int_{\Gamma_j'} F(\kappa)[L'(\kappa',\zeta')^{-1} \oplus 0_{(m-1)n}]E(\kappa,\zeta)\tilde{A}_1(\zeta) d\kappa' .
\end{aligned}
\tag{8.9}
$$

Now it is obvious that the projectors $P_j^{(1)}(\zeta')$ depend analytically on $\zeta'$ for $z' \neq 0$.

Lemma 8.1. The projectors $P_k(\zeta)$, $k = 2,3,\ldots,n$, depend analytically on $\zeta \in \Omega(\zeta_0)$.

Proof: The matrix $C(\kappa,\xi_0)$ is simply $(\kappa-1)A$. Therefore there exists a matrix $D(\kappa,\xi)$ invertible and analytic in the neighbourhood $\Omega_0(\kappa_k) x \Omega_0(\zeta_0)$ such that

$$
D^{-1}(\kappa,\xi)C(\kappa,\xi)D(\kappa,\xi) = \mathrm{diag}(0,c_2,c_3,\ldots,c_n)
\tag{8.10}
$$

where the eigenvalues $c_i = c_i(\kappa,\xi)$, $i = 2,3,\ldots,n$, depend analytically on $\kappa$ and $\zeta$, and $c_i(\kappa,\zeta_0) = (\kappa-1)a_i$. Then

$$
D^{-1}(\kappa,\xi)L^{-1}(\kappa,\zeta)D(\kappa,\xi) = \mathrm{diag}[\kappa(z-1),\ \ell(c_2,\kappa,\zeta),\ldots,\ell(c_n,\kappa,\zeta)]^{-1}
\tag{8.11}
$$

where the polynomial $\ell(c,\kappa,\zeta)$ is defined as in subsection 6.2. For $z \neq 1$ all the diagonal elements in (8.11) except $\ell(c_k,\kappa,\zeta)^{-1}$ are analytic in $\Omega(\kappa_k)$ as functions of $\kappa$, and $\ell(c_k,\kappa,\zeta)^{-1}$ is analytic in $\Gamma_k x \Omega(\zeta_0)$ as a function of $\kappa$ and $\zeta$. Therefore

$$
\begin{aligned}
P_k(\zeta) = (2\pi i)^{-1} \oint F(\kappa)[&D(\kappa,\xi)\mathrm{diag}(0,0,\ldots,\ell(c_k,\kappa,\zeta)^{-1},0,\ldots,0)D^{-1}(\kappa,\xi) \\
&\oplus I_{(m-1)n}]E(\kappa,\zeta)\tilde{A}_1(\zeta) d\kappa
\end{aligned}
$$

and the analyticity of $P_k(\zeta)$ in $\Omega(\zeta_0)$ is proved.

The projectors $P_0(\zeta)$ and $P_\infty(\zeta)$ are not analytic as z tends to one. However, the following lemma holds:

Lemma 8.2. a) There exist matrix valued functions $X_0(\zeta)$ and $X_\infty(\zeta) = (X_\infty^{(1)}(\zeta), X_\infty^{(2)}(\zeta))$ analytic in $\Omega(\zeta_0)$, the columns of which are independent for any $\zeta\in\Omega(\zeta_0)$ and form for $z \neq 1$ a basis of the spaces Im $P_0(\zeta)$ and Im $P_\infty(\zeta)$ respectively. b) $X_0(\zeta)$ is one column matrix and consists of the singular eigenvector $\tilde{\varphi}_0(0,\xi)$. The columns of $X_\infty^{(1)}(\xi,z)$ form a singular Jordan chain of length m-1 corresponding to the eigenvalue $\kappa = 0$ of $\tilde{L}^{(\infty)}(\kappa,\xi,1)$; this chain is generated by the singular root function $\tilde{\varphi}_0^{(\infty)}(\kappa,\xi)$ at the point $\kappa = 0$.

c) The columns of the matrix $(X_0(0,z),X_\infty^{(1)}(0,z))$ form a basis of the space Ker $\tilde{A}$, where $\tilde{A} = \mathrm{diag}(A,A,\dots,A)$. The columns of $X_\infty^{(2)}(\zeta_0)$ form a basis of the space Im $\mathrm{diag}(0,0,A,A,\dots,A)$ and are independent of the space Ker $\tilde{P}(0)$.

d) There are matrix valued functions $M_0(\zeta) \equiv 0$ and $M_\infty(\zeta) = M_\infty^{(1)}(\zeta)\oplus M_\infty^{(2)}(\zeta)$ analytic in $\Omega(\zeta_0)$ and satisfying the identities

$$\begin{aligned}\tilde{A}_1(\zeta)X_0(\zeta)M_0(\zeta) + \tilde{A}_0(\zeta)X_0(\zeta) &= 0\\ \tilde{A}_0(\zeta)X_\infty(\zeta)M_\infty(\zeta) + \tilde{A}_1(\zeta)X_\infty(\zeta) &= 0\end{aligned}\tag{8.12}$$

and $M_\infty(\zeta)$ is a Jordan matrix with eigenvalue $\kappa = 0$.

Proof: We consider only the case $k = \infty$ since the case $k = 0$ is analogous to the first one for $m = 2$. As in (8.10), (8.11) we have a similarity transformation of the matrices $C^{(\infty)}(\kappa,\xi)$ and $L^{(\infty)}(\kappa,\zeta)^{-1}$ in the neighbourhood $\Omega_0(\kappa_0)\mathrm{x}\Omega(\zeta_0)$. Let us partition the corresponding matrix $D(\kappa,\xi)=(D_1(\kappa,\xi),D_2(\kappa,\xi))$, where $D_1(\kappa,\xi)$ is the first column and $D_2(\kappa,\xi)$ consists of the remaining n-1 columns of $D(\kappa,\xi)$. We may suppose that $D_1(\kappa,\xi) = \varphi_0(-\alpha(\kappa,\xi),\beta(\kappa,\xi))$ so that

$F_m^{(\infty)}(\kappa)D_1(\kappa,\xi) = \tilde{\varphi}_0^{(\infty)}(\kappa,\xi)$. The columns of $D_2(\kappa,0)$ are the eigenvectors of A corresponding to the non-zero eigenvalues $a_2,a_3,\dots,a_n$ and span the space Im A. We may assume that the matrix $D_2(\kappa,0)$ does not depend on $\kappa$.

There is a following factorization

$$(\kappa^{m-2}L^{(\infty)}(\kappa,\zeta))^{-1} = [D(\kappa,\xi)\mathrm{diag}(\kappa^{1-m},\kappa^{2-m},\dots,\kappa^{2-m})]$$

$$\cdot[D(\kappa,\xi)\mathrm{diag}(z-1,\ell(c_2,\kappa_1\zeta),\dots,\ell(c_n,\kappa_1\zeta))]^{-1}$$

where the second matrix in the product is analytic in $\Omega(\kappa_0)$ for $z \neq 1$. Hence for $z \neq 1$ the projector $P_\infty(\zeta)$ may be replaced by an operator

$$(8.13)\quad Q_\infty(\zeta)\varphi$$

$$= (2\pi i)^{-1}\oint_{\Gamma_0} F^{(\infty)}(\kappa)[D(\kappa,\xi)\mathrm{diag}(\kappa^{1-m},\kappa^{2-m},\dots,\kappa^{2-m})\oplus I_{(m-1)n}]\varphi(\kappa)d\kappa$$

which acts on the space $\Phi(\Omega(\kappa_0))$ with values in $\mathbb{C}^{mn}$. The operator $Q_\infty(\zeta)$ depends only on $\xi$, is analytic in $\Omega(\zeta_0)$ and has for $z \neq 1$ the same image as $P_\infty(\zeta)$. Let us define operators

$$(8.14)\quad Q_\infty^{(1)}(\zeta)\varphi = (2\pi i)^{-1}\oint_{\Gamma_0} F^{(\infty)}(\kappa)[D(\kappa,\xi)\mathrm{diag}(\kappa^{1-m},1,1,\dots,1)\oplus I_{(m-1)n}]\varphi(\kappa)d\kappa$$

$$= (2\pi i)^{-1}\oint_{\Gamma_0} F_m^{(\infty)}(\kappa)D_1(\kappa,\xi)\kappa^{1-m}\varphi^{(1)}(\kappa)d\kappa$$

and

$$(8.15)\quad Q_\infty^{(2)}(\zeta)\varphi=(2\pi i)^{-1}\oint_{\Gamma_0} F^{(\infty)}(\kappa)[D(\kappa,\xi)\mathrm{diag}(1,\kappa^{2-m},\dots,\kappa^{2-m})\oplus I_{(m-1)n}]\varphi(\kappa)d\kappa$$

$$=(2\pi i)^{-1}\oint_{\Gamma_0} F_m^{(\infty)}(\kappa)D_2(\kappa,\xi)\kappa^{2-m}\varphi^{(2)}(\kappa)d\kappa$$

where $\varphi^{(1)}(\kappa)$ is the first component of $\varphi(\kappa)$ and $\varphi^{(2)}(\kappa)$ consists of the next n-1 components of $\varphi(\kappa)$. Obviously $Q_\infty(\zeta) = Q_\infty^{(1)}(\zeta) + Q_\infty^{(2)}(\zeta)$ for any $\zeta \in \Omega(\zeta_0)$, and as in lemma 7.1 one can prove that for $z \neq 1$ the space Im $Q_\infty(\zeta)$ is the direct sum of the spaces Im $Q_\infty^{(1)}(\zeta)$ and Im $Q_\infty^{(2)}(\zeta)$ of dimensions m-1 and (m-2)(n-1) respectively. Since $Q_\infty(\zeta)$, $Q_\infty^{(1)}(\zeta)$ and $Q_\infty^{(2)}(\zeta)$ do not depend on z, the above statement is also true for z=1. Taking $\varphi^{(1)}(\kappa)$ in (8.14) equal correspondingly to $\kappa^{m-1}, \kappa^{m-2}, \ldots, \kappa, 1$ we obtain the columns of the matrix $X_\infty^{(1)}(\zeta)$, which form a Jordan chain of length m-1 generated by the singular root function $\tilde{\varphi}_0^{(\infty)}(\kappa,\xi)$ at the point $\kappa=0$. These columns obviously form a basis of Im $Q_\infty^{(1)}(\zeta)$. Since for $\xi=0$ $\varphi_0(\alpha,\beta) = \varphi_0(\alpha,0) \in \text{Ker}A$, it is easy to show that

$$\text{Im } Q_\infty^{(1)}(0,z) = \text{Ker diag}(I,A,A,\ldots,A).$$

Similarly, taking $\varphi^{(2)}(\kappa)$ in (8.15) equal to

$(\kappa^{m-2-k},0,\ldots,0)'$, $(0,\kappa^{m-2-k},\ldots,0)',\ldots,(0,0,\ldots,\kappa^{m-2-k})'$, where $k=1,2,\ldots,m-2$,

we obtain (m-2)(n-1) columns of the matrix $X_\infty^{(2)}(\zeta)$, which form a basis of the space Im $Q_\infty^{(2)}(\zeta)$. Thus, the columns of $X_\infty(\zeta) = (X_\infty^{(1)}(\zeta), X_\infty^{(2)}(\zeta))$ form a basis of the space Im $Q_\infty(\zeta)$ for any $\zeta \in \Omega(\zeta_0)$ and, therefore, also a basis of Im $P_\infty(\zeta)$ for $z \neq 1$. Since the vector $X_0(0,z) = \tilde{\varphi}_0(0,0)$ spans the space Ker diag(A,I,I,...,I), the columns of $(X_0(0,z), X_\infty^{(1)}(0,z))$ form a basis of Ker $\tilde{A}$. The columns of $X_\infty^{(2)}(\zeta)$ form a Jordan sequence of $\tilde{L}^{(\infty)}(\kappa,\zeta)$ corresponding to the eigenvalue $\kappa=0$. This Jordan sequence is generated by n-1 root functions, which are columns of the matrix $F_m^{(\infty)}(\kappa)D_2(\kappa,\xi)$. Since the columns of $D_2(\kappa,0)$ form a basis of Im A and do not depend on $\kappa$, it is easy to show that the columns of $X_\infty^{(2)}(\zeta_0)$ form a basis of the space Im diag(0,0,A,A,...,A). Let us recall that Ker $\tilde{P}(0) = F_1(1)V_0 + \text{Ker}\,\tilde{A}$. If a vector $\tilde{\varphi} = (\varphi_1, \varphi_2, \dots, \varphi_m)'$ (here $\varphi_1, \varphi_2, \dots, \varphi_m$ are n-dimensional vectors) belongs to Ker $\tilde{P}(0) \cap \text{Im diag}(0,0,A,A,\dots,A)$, then the "component vectors" $\varphi_1$ and $\varphi_2$ are zero and therefore $\tilde{\varphi} \in \text{Ker}\,\tilde{A}$. Since Ker $A \cap \text{Im}\,A = 0$, it follows that $\tilde{\varphi} = 0$, so that part c) of the lemma is also proved. Since for any $\zeta \in \Omega_0(\zeta_0)$ the matrices $X_\infty^{(1)}(\zeta)$ and $X_\infty^{(2)}(\zeta)$ consist of Jordan chains, the matrix $M_\infty^{(1)}(\zeta)$ is a single Jordan cell and $M_\infty^{(2)}(\zeta)$ is a direct sum of n-1 Jordan cells of order m-2 with the eigenvalue $\kappa = 0$. The identities in (8.12) follow now immediately.

Let us now study the projectors $P_j^{(1)}(\zeta')$. Let $z_0' \neq 0$. Then $P_j^{(1)}(\zeta')$ is analytic in $\Omega(\zeta_0')$ and the image of $P_j^{(1)}(\zeta')$ has a constant dimension $q_j^{(1)}$. The projector $P_j^{(1)}(\zeta')$ may be replaced by an operator $Q_j^{(1)}(\zeta') : \Phi(\Omega(\kappa_j')) \longrightarrow C^{mn}$ given by

$$Q_j^{(1)}(\zeta')\varphi = (2\pi i)^{-1} \oint_{\Gamma_j'} F_1(\kappa) L'(\kappa', \zeta')^{-1} \varphi(\kappa') d\kappa' . \tag{8.16}$$

For $r \neq 0$ the vector function $\varphi(\kappa')$ depends analytically also on $\kappa = 1 + r\kappa'$ in a neighbourhood of the point $1 + r\kappa_j$, and therefore the images of $Q_j^{(1)}(\zeta')$ and $P_j^{(1)}(\zeta')$ coincide. Since $Q_j^{(1)}(\zeta')$ is analytic in $\Omega(\zeta_0')$ and for $r = 0$ obviously $\text{Im}\,Q_j^{(1)}(\zeta') \supseteq \text{Im}\,P_j^{(1)}(\zeta')$, it follows that $\text{Im}\,Q_j^{(1)}(\zeta') = \text{Im}\,P_j^{(1)}(\zeta')$ for any $\zeta' \in \Omega(\zeta_0')$. Therefore one can define in Im $P_j^{(1)}(\zeta')$ a basis, which depends analytically on $\zeta'$ and whose vectors are columns of a matrix $X_j^{(1)}(\zeta')$ given by

$$X_j^{(1)}(\zeta') = Q_j^{(1)}(\zeta')(\Psi(\kappa')),$$

where $\Psi(\kappa')$ is a $n \times q^{(1)}$ matrix analytic in $\Omega(\kappa_j')$. Since the integrand in (8.16) being multiplied on the left by $\tilde{L}(\kappa, \zeta)$ becomes an analytic function in $\Omega(\kappa_j')$, we obtain

$$\tilde{A}_1(\zeta)Q_j^{(1)}(\zeta')(\kappa\Psi(\kappa')) + \tilde{A}_0(\zeta)X_j^{(1)}(\zeta') = 0 \ .$$

Expressing $Q_j^{(1)}(\zeta')(\kappa'\Psi(\kappa'))$ in the basis $X_j^{(1)}(\zeta')$ as $X_j^{(1)}(\zeta')M_j'(\zeta')$, where $M_j'(\zeta')$ is analytic in $\Omega(\zeta_0')$, we arrive at the identity

(8.17) $\tilde{A}_1(\zeta)X_j^{(1)}(\zeta')M_j^{(1)}(\zeta')+\tilde{A}_0(\zeta)X_j^{(1)}(\zeta')=0$, where $M_j^{(1)}(\zeta')=I+rM_j'(\zeta')$ .

The characteristic equation $|\kappa I-M_j^{(1)}(\zeta')| = |r(\kappa' I-M_j'(\zeta'))| = 0$ has for $r\neq 0$ the same $\kappa'$-roots in $\Omega(\kappa_j')$ as equation (8.6). It follows from the continuity considerations that the equations $|\kappa' I-M_j'(\zeta')|$ and (8.6) are equivalent in $\Omega(\kappa_j')$ also for $r = 0$, and therefore the matrix $M_j'(\zeta_0')$ has the eigenvalue $\kappa_j'$ of multiplicity $q_j^{(1)}$.

In the next subsection we shall need the following

Lemma 8.3. Let $\text{Re}\ z_0' = 0$, $z_0' \neq 0$ and $\text{Re}\ \kappa_j' = 0$. Then the matrix $M_j'(\zeta_0')$ has only one eigenvector corresponding to the eigenvalue $\kappa_j'$.

Proof: The operator $Q_j^{(1)}(\zeta')$ for $\zeta' = \zeta_0'$ may be written in a form

$$Q_j^{(1)}(\zeta_0')\varphi = (2\pi i)^{-1}F_1(1)\oint_{\Gamma_j'} L'(\kappa',\zeta_0')^{-1}\varphi(\kappa')d\kappa' \ .$$

Let us denote by $Q_j'(\zeta_0)$ the operator from $\Phi(\Omega(\kappa_j'))$ to $\mathbb{C}^n$ represented by the above integral. Recall that $L'(\kappa',\zeta_0') = z_0'I + A\kappa' + Bi\xi_0'$ is a linear regular $\kappa'$-matrix. From the strict hyperbolicity we conclude that this matrix has only one eigenvector corresponding to the eigenvalue $\kappa' = \kappa_j'$. Let $v$ be an eigenvector of $M_j'(\zeta_0')$. By the equality $Q_j^{(1)}(\kappa'\Psi(\kappa')) = Q_j^{(1)}(\Psi(\kappa'))M_j'$ we obtain $Q_j^{(1)}((\kappa'-\kappa_j')\Psi(\kappa')v) = 0$. But then also $Q_j'((\kappa'-\kappa_j')\Psi(\kappa')v) = 0$. From (2.15) we get for any $\varphi\in\Phi(\Omega(\kappa_j'))$ an identity

$$L'(\kappa_j',\zeta_0')Q_j'(\varphi(\kappa')) = AQ_j'((\kappa_j'-\kappa')\varphi(\kappa')) \ .$$

Therefore $Q_j'(\Psi(\kappa')v) = Q_j'(\Psi(\kappa'))v$ is an eigenvector of $L'(\kappa',\zeta_0')$ corresponding to the eigenvalue $\kappa' = \kappa_j'$. Let us note that since the columns of $Q_j^{(1)}(\Psi(\kappa'))$ are independent, so are the columns of $Q_j'(\Psi(\kappa'))$. Hence the vector v is unique, and the lemma is proved.

Let us now investigate the projectors $P_j^{(1)}(\zeta')$ in $\Omega(\zeta_0')$ when $\underline{z_0' = 0}$.

Lemma 8.4. a) There exists a matrix valued function $X_j^{(1)}(\zeta')$, $j = 1,2,\dots,t$, analytic in $\Omega(\zeta_0')$, the columns of which are independent for any $\zeta'\in\Omega(\zeta_0')$ and form a basis of Im $P_j^{(1)}(\zeta')$ when $z' \neq 0$.

b) For $z' = 0$ the columns of $X_j^{(1)}(\zeta')$ belong to Ker $\tilde{P}(\xi)$ (where $\xi = \xi'\cdot r$) and $X_j^{(1)}(\zeta_0') = F_1(1)X_j'(\zeta_0')$, where the columns of $X_j'(\zeta_0')$ form a singular Jordan chain of length $q_j^{(1)}$ corresponding to the eigenvalue $\kappa' = \kappa_j'$ of the singular $\kappa'$-matrix $L'(\kappa',\zeta_0') = A\kappa' + Bi\xi_0'$.

c) There is a matrix valued function $M_j'(\zeta')$ of order $q_j^{(1)} x q_j^{(1)}$ analytic in $\Omega(\zeta_0')$ such that the identity (8.17) is valid. The matrix $M_j'(\zeta_0')$ is a Jordan cell with the eigenvalue $\kappa_j'$.

Proof: Using the factorization in (8.5) and taking into account the fact that the matrix $(s_2I+C)^{-1}$ depends analytically on $\kappa'$ in $\Omega(\kappa_j')$, we replace the projector $P_j^{(1)}(\zeta')$ in (8.9) by an operator $Q_j^{(1)}(\zeta') : \Phi(\Omega(\kappa_j')) \longrightarrow \mathbb{C}^{mn}$ given by

$$Q_j^{(1)}(\zeta')\varphi = (2\pi i)^{-1} \oint_{\Gamma_j'} F_1(\kappa)(s_1'\cdot T+C')^{-1}\varphi(\kappa')d\kappa' \ . \tag{8.18}$$

For $rz' \neq 0$ the operator $Q_j^{(1)}(\zeta')$ has the same image as the projector $P_j^{(1)}(\zeta')$.

If $r = 0$ but $z' \neq 0$, both $Q_j^{(1)}(\zeta')$ and $P_j^{(1)}(\zeta')$ are analytic, and it follows as in the case $z_0' \neq 0$ that $\operatorname{Im} Q_j^{(1)}(\zeta') = \operatorname{Im} P_j^{(1)}(\zeta')$. We proceed as in lemma 3.4. The operator in (8.18) is replaced by a new one, denoted by the same letter

$$(8.19)\quad Q_j^{(1)}(\zeta')\varphi = (2\pi i)^{-1} \oint_{\Gamma_j'} F_1(\kappa)D(\alpha',\beta')[(N_0'(\kappa',\zeta'))^{-1} \oplus 0_{n-\rho}]\varphi(\kappa')d\kappa' .$$

The matrix $N_0'(\kappa',\zeta')$ is given as in (3.28), where $\lambda',\omega'$ and $s'$ should be replaced by $\alpha',\beta'$ and $z'$ respectively and

$$N_0'(\kappa',\zeta_0') = \operatorname{diag}((\kappa'-\kappa_j')^{q_j^{(1)}},1,1,\dots,1) .$$

The first column of the matrix $D(\alpha',\beta')$ is the singular root function $\varphi_0(\alpha',\beta')$ and is proportional for $r \neq 0$ to the vector $\varphi_0(\alpha,\beta)$. According to (6.26) $F_1(\kappa)\varphi_0(\alpha,\beta) = \tilde{\varphi}_0(\kappa,\zeta)$ so that the first column of the matrix $F_1(\kappa)D(\alpha',\beta')$ belongs to the space $\tilde{V}_0(\xi) = \operatorname{Ker} \tilde{P}(\xi)$, where $\xi = \xi'\cdot r$. If $r = 0$, $\varphi_0(\alpha',\beta') = \varphi_0(\kappa',i\xi')$ and

$$F_1(\kappa)\varphi_0(\alpha',\beta') = F_1(1)\varphi_0(\alpha',\beta') \in F_1(1)V_0 \subset \operatorname{Ker}\tilde{P}(0).$$

For $z' = 0$ it follows then from the diagonal form of $N_0'(\kappa',\zeta')$ that $\operatorname{Im} Q_j^{(1)}(\zeta') \subset \operatorname{Ker} \tilde{P}(\xi)$. Let us define the matrix $\Psi(\kappa')$ as $\Psi(\lambda')$ in lemma 3.4. The matrix $X_j^{(1)}(\zeta')$ is determined now by $X_j^{(1)}(\zeta') = Q_j^{(1)}(\zeta')(\Psi(\kappa'))$. For $\zeta' = \zeta_0'$ we have

$$X_j^{(1)}(\zeta_0') = F_1(1)X_j'(\zeta_0'),$$

where

$$X_j'(\zeta') = (2\pi i)^{-1}\oint_{\Gamma_j'} \varphi_0(\kappa',i\zeta_0')\operatorname{diag}((\kappa'-\kappa_j')^{-q_j^{(1)}},0,0,\dots,0)\Psi(\kappa')d\kappa'$$

so that $X_j'(\zeta_0')$, as claimed in part b) of the lemma, is a singular Jordan chain generated by the root function $\varphi_0(\kappa',i\xi'_0)$ at the point $\kappa' = \kappa'_j$. As in the differential case $q_j^{(1)} \leq (n-1)/2$. According to assumption 1.2 and lemma 2.1 the columns of $X'_j(\zeta'_0)$ and therefore of $X_j^{(1)}(\zeta'_0)$ are independent. We shall choose $\Omega(\zeta'_0)$ small enough such that the columns of $X_j^{(1)}(\zeta')$ are independent for any $\zeta'\in\Omega(\zeta'_0)$. Since the image of $Q_j^{(1)}(\zeta')$ in (8.19) coincides with the one of $P_j^{(1)}(\zeta')$ for $z' \neq 0$ and has dimension $q_j^{(1)}$, it follows that the columns of $X_j^{(1)}(\zeta')$ form a basis of Im $Q_j^{(1)}(\zeta')$ for any $\zeta'\in\Omega(\zeta'_0)$. To obtain the matrix $M_j^{(1)}(\zeta')$ and formula (8.17) we proceed as in the case $z'_0 \neq 0$. The Jordan form of the matrix $M'_j(\zeta'_0)$ follows immediately from the definition of $\Psi(\kappa')$ and diagonal form of the matrix $N'_0(\kappa',\zeta'_0)$.

We are now able to bring the $\kappa$-matrix $L(\kappa,\zeta)$ to a block form. In addition to the already defined matrices $X_0(\zeta)$, $X_\infty(\zeta)$ and $X_j^{(1)}(\zeta')$ we determine matrix $X_k(\zeta)$, $k = 2,3,\ldots,n$, analytic in $\Omega(\zeta_0)$, the columns of which form a basis of the space Im $P_k(\zeta)$. To the matrix $X_k(\zeta)$ corresponds a square matrix $M_k(\zeta)$ analytic in $\Omega(\zeta_0)$ such that

$$\tilde{A}_1(\zeta)X_k(\zeta)M_k(\zeta) + \hat{A}_0(\zeta)X_k(\zeta) = 0 . \tag{8.20}$$

Since $\kappa_k = (a_k+1)/(a_k-1)$ is a simple root, the matrix $X_k(\zeta)$ is actually an eigenvector of the $\kappa$-matrix $\tilde{L}(\kappa,\zeta)$ and $M_k(\zeta_0) = \kappa_k$. We shall often consider the matrices $X_k(\zeta)$ and $M_k(\zeta)$, $k = 0,2,3,\ldots,n,\infty$, as functions of $\zeta'$ through the relation in (8.2). Let us denote

$$\begin{aligned} &X_{F1}^{(1)} = (X_1^{(1)},X_2^{(1)},\ldots,X_t^{(1)}), \quad X_{F1} = (X_{F1}^{(1)},X_2,X_3,\ldots,X_n) \\ (8.21)\qquad & \\ &X_F = (X_0,X_{F1}), \quad X = (X_F,X_\infty) . \end{aligned}$$

in a neighbourhood $\Omega(\zeta_0')$ with $z_0' = 0$ we partition additionally

(8.22) $X = (X^{(1)},X^{(2)})$, where $X^{(1)} = (X_0,X_{F1}^{(1)},X_\infty^{(1)})$, $X^{(2)} = (X_2,X_3,\dots,X_n,X_\infty^{(2)})$.

The eigenvalues $\kappa_k$, $k = 2,3,\dots,n$, split up into two groups I and II, according to whether $|\kappa_k| < 1$ or $|\kappa_k| > 1$. In the case Re $z_0' > 0$ or $z_0' = 0$ we split in the same way the eigenvalues $\kappa_j'$, $j = 1,2,\dots,t$, according to whether Re $\kappa_j' < 0$ or Re $\kappa_j' > 0$. Then the matrix $X_F$ is also partitioned as $X_F = (X_0,X_I,X_{II})$. If $z_0' = 0$ we suppose the matrices $X_I$ and $X_{II}$ to be partitioned as $X_I = (X_I^{(1)},X_I^{(2)})$ and $X_{II} = (X_{II}^{(1)},X_{II}^{(2)})$. We construct also a block matrix $M_F$, which corresponds to $X_F$ and is partitioned according to $X_F$ with the similar notations for the partial matrices. As usual, introduce the matrix $T = (A_1X_F,A_0X_\infty)$. The rows of the inverse matrix $T^{-1}$ are partitioned and denoted so that they correspond to the columns of X. Introducing the matrices $B_0$ and $B_1$ given in (7.42) we rewrite the identities (8.12), (8.17) and (8.20) as

$$\hat{L}(\kappa,\zeta)X(\zeta') = T(\zeta')(\hat{B}_0(\zeta') + \kappa B_1(\zeta')) , \tag{8.23}$$

where $\zeta' \in \Omega_0(\zeta_0')$ and $\zeta$ is connected with $\zeta'$ by (8.2).

Let us first investigate the matrices $X(\zeta')$ and $T^{-1}(\zeta')$ in the case $z_0' \neq 0$. Denote $\hat{T}^{-1}(\zeta') = rT^{-1}(\zeta')$ and partition $\hat{T}^{-1}$ according to $T^{-1}$.

Lemma 8.6. a) The matrix $X(\zeta')$ is invertible in $\Omega(\zeta_0')$ and $T^{-1}$ is analytic in $\Omega(\zeta_0')$.

b) Moreover, the matrices $(T^{-1}(\zeta'))_{F1}$, $(T^{-1}(\zeta'))_\infty^{(2)}$ are analytic in $\Omega(\zeta_0')$, and the last row of $(\hat{T}^{-1}(\zeta'))_\infty^{(1)}$ is non-zero.

Proof. Let $X(\zeta_0')v = 0$. We suppose that the components of the vector v are partitioned and denoted according to the matrix X. Let us recall that the projectors $P_k(\zeta')$, $k = 2,3,\dots,n$, and $P_j^{(1)}(\zeta')$, $j = 1,2,\dots,t$, are mutually orthogonal and analytic in $\Omega(\zeta_0')$ and vanish on the spaces Im $Q_0(\zeta)$ and Im $Q_\infty(\zeta)$. Therefore we get immediately that $v_{F1} = 0$ and $X_0(\zeta_0)v_0 + X_\infty(\zeta_0)v_\infty = 0$. According to part c) of lemma 8.2 the columns of $(X_0(\zeta_0), X_\infty^{(1)}(\zeta_0))$ form a basis of Ker $\tilde{A} \subset$ Ker $\tilde{P}(0)$, and the columns of $X_\infty^{(2)}(\zeta_0)$ are independent of Ker $\tilde{P}(0)$. Hence $v = 0$, and the matrix $X(\zeta_0')$ is invertible. We can choose $\Omega(\zeta_0')$ small enough so that $X(\zeta')$ is invertible for any $\zeta' \in \Omega(\zeta_0')$. Let us fix any $\kappa$ with $|\kappa| = 1$. From stability of the Cauchy problem we have for any $\zeta' \in \Omega_R(\zeta_0')$ an estimate

$$\| X(\zeta')(\tilde{B}_0(\zeta')+\kappa\tilde{B}_1(\zeta'))^{-1}T^{-1}(\zeta')\| = \|\tilde{L}^{-1}(\kappa,\zeta)\| \le \frac{K}{|z|-1} . \tag{8.23}$$

Since $X(\zeta')$ is invertible and $\tilde{B}_1(\zeta') + \kappa\, \tilde{B}_1(\zeta')$ is bounded, it follows that $\|T^{-1}(\zeta')\| \le \frac{K}{|z|-1}$ and $\|\hat{T}^{-1}(\zeta')\| \le \frac{Kr}{|z|-1}$. The matrix $\hat{T}^{-1}(\zeta')$ has a singularity of the type $|T(\zeta')|^{-1}$. Since the matrix $T(\zeta')$ is invertible for $r \neq 0$ and $|T(\zeta')| = 0$ if $r = 0$, the matrix $\hat{T}^{-1}(\zeta')$ may be written as a fraction $\hat{T}^{-1}(\zeta') = \Psi(\zeta')/r^k$, where the matrix $\Psi(\zeta')$ is analytic in $\Omega(\zeta_0')$. If the component $z'$ in $\zeta' = (\xi', z', r) \in \Omega_R(\zeta_0')$ is fixed and Re $z' > 0$, the matrix $\hat{T}^{-1}(\zeta')$ is bounded as $r \to 0$, and the above fraction is reducible. Therefore this fraction is reducible for any $\zeta' \in \Omega(\zeta_0')$, and $\hat{T}^{-1}(\zeta')$ is analytic in $\Omega(\zeta_0')$.

Let $r = 0$. As in lemma 3.6 we have Im $\hat{T}^{-1}(\zeta')$ = Ker $T(\zeta')$. Let $\kappa$ in (8.23) be fixed and different from all the eigenvalues of $\tilde{B}_0(\zeta')+\kappa\tilde{B}_1(\zeta')$ for all $\zeta' \in \Omega(\zeta_0')$. If $v \in$ Ker $T(\zeta')$, then $\tilde{L}(\kappa,\zeta_0)X(\zeta')(\tilde{B}_0(\zeta')+\kappa\tilde{B}(\zeta'))^{-1}v = 0$. Denoting $u = (\tilde{B}_0(\zeta')+\tilde{B}_1(\zeta'))^{-1}v$ we obtain that $X(\zeta')u \in$ Ker $\tilde{L}(\kappa,\zeta_0)$. We suppose the components of the vectors u and v to be partitioned and denoted according to the matrix X. The matrix $\tilde{L}(\kappa,\zeta_0)$ is singular of order one with singular root function $\varphi(\kappa,z) = F_1(\kappa)\varphi_1(1,z)$, which span the singular eigenspace $V_1(\kappa)$ = Ker A. As in lemma [illegible] we can show that the kernel of $\tilde{L}(\kappa,\zeta_0)$ is spanned by the vector $\varphi_1(\kappa,0)$.

Since the columns of $(X_0(\zeta_0),\ X_\infty^{(1)}(\zeta_0))$ form a basis of Ker $\tilde{A}$ and the remaining columns of $X(\zeta')$ are independent of Ker $\tilde{A}$, it follows that $u_{F1}(\zeta') = u_\infty^{(2)}(\zeta')=0$ and

$$X_0(\zeta_0)u_0(\zeta') + X_\infty^{(1)}(\zeta_0)u_\infty^{(1)}(\zeta') \sim \tilde{\varphi}_0(\kappa,0).$$

Since the vectors $\tilde{\varphi}_0(\kappa,0)$ for different $\kappa$ span the space Ker $\tilde{A}$ , we may assume that the last component of $u_\infty^{(1)}(\zeta')$ is different from zero. The components $v_{F1}(\zeta')$ and $v_\infty^{(1)}(\zeta')$ are given by $v_{F1}(\zeta') = (\kappa I-M_{F1}(\zeta'))u_{F1}(\zeta') = 0$ and $v_\infty^{(2)}(\zeta') = (I-\kappa M_\infty^{(2)}(\zeta_0))u_\infty^{(2)}(\zeta') = 0$. Therefore $(\hat{T}^{-1}(\zeta'))_{F1} = (\hat{T}^{-1}(\zeta'))_\infty^{(2)}=0$ and the matrices $(T^{-1}(\zeta'))_{F1}$ , $(T^{-1}(\zeta'))_\infty^{(2)}$ are analytic in $\Omega(\zeta_0')$. Since $v_\infty^{(1)}(\zeta') = (I-\kappa M_\infty^{(1)}(\zeta_0))u_\infty^{(1)}(\zeta')$ and the matrix $M_\infty^{(1)}(\zeta_0)$ is a nilpotent Jordan cell, we conclude that the last component of $v_\infty^{(1)}(\zeta')$ is non-zero. Taking $\zeta' = \zeta_0'$ we obtain finally that the last row of the matrix $(\hat{T}^{-1}(\zeta_0'))_\infty^{(1)}$ is non-zero, and the lemma is completely proved.

Let us now investigate the matrices $X(\zeta')$ and $T^{-1}(\zeta')$ in the case $\underline{z_0' = 0}$.

Lemma 8.6. a) The matrix $X(\zeta')$ is non-singular for $z' \neq 0$.

b) For $\zeta' = (\xi',0,r)\in\Omega(\zeta_0')$ the columns of $X^{(1)}(\zeta')$ belong to the space Ker $\tilde{P}(\xi)$, where $\xi = \xi' r$, and the columns of $(X_0(\zeta'),X_I^{(1)}(\zeta'),X_\infty^{(1)}(\zeta'))$ as well as the columns of $(X_0(\zeta'),X_{II}^{(1)}(\zeta'),X_\infty^{(1)}(\xi'))$ form a basis of Ker $\tilde{P}(\xi)$.

c) The columns of $X^{(2)}(\zeta')$ are independent of Ker $\tilde{P}(\xi)$. Hence the matrix $(X_0(\zeta'),X_I(\zeta'))$ is of full rank $n$.

Proof: Part a) *of the lemma* follows as in lemma 8.5.

According to part b) of lemmas 8.2 and 8.4 the columns of $X^{(1)}(\zeta')$ belong for $z' = 0$ to the space Ker $\tilde{P}(\xi)$. Furthermore, the columns of $(X_0(\zeta_0'),X_\infty^{(1)}(\zeta_0'))$ form a basis of Ker $\tilde{A}$ and $X_{F1}^{(1)}(\zeta_0') = P_1(1)X_{F1}'(\zeta_0')$, where the matrix $X'_{F1}(\zeta'_0) = (X'_1(\zeta'_0),\dots,X'_t(\zeta'_0))$ consists of singular Jordan chains of the $\kappa'$-matrix $L'(\kappa',\zeta'_0) = A\kappa' + Bi\xi'_0$. Let us partition the matrix $X'_{F1}(\zeta'_0)$ according to the matrix $X^{(1)}_{F1}(\zeta'_0)$ as $X'_{F1}(\zeta'_0) = (X'_I(\zeta'_0), X'_{II}(\zeta'_0))$. Then the $(n-1)/2$ columns of $X'_I(\zeta_0)$ together with the vector $\varphi_0(1,0)\in\operatorname{Ker} A$ form a basis of the $(n+1)/2$ dimensional space $V_0$. Therefore the columns of $(X_0(\zeta'_0),\ X^{(1)}_I(\zeta'_0),\ X^{(1)}_\infty(\zeta'_0))$ form a basis of the space $\operatorname{Ker}\tilde{P}(0) = \operatorname{Ker}\hat{A} + F_1(1)V_0$. From the consideration of continuity the last statement remains true if $\zeta'_0$ is replaced by any $\zeta' = (\xi',0,r)\in\Omega(\zeta'_0)$ and $\operatorname{Ker}\tilde{P}(0)$ by $\operatorname{Ker}\tilde{P}(\xi)$, where the neighbourhood $\Omega(\zeta'_0)$ is sufficiently small. In the same way one considers the matrix $(X_0(\zeta'),\ X^{(1)}_{II}(\zeta'),\ X^{(1)}_\infty(\zeta'))$.

The matrices $M^{(1)}_I(\zeta'_0) = I$, $M^{(2)}_F(\zeta'_0)$ and $M^{(2)}_\infty(\zeta'_0)$ are in Jordan form and therefore the columns of $(X^{(1)}_I(\zeta'_0),\ X^{(2)}_F(\zeta'_0), X^{(2)}_\infty(\zeta'_0))$ form a Jordan sequence of the $\kappa$-matrix $\tilde{L}(\kappa,\zeta_0)$. The columns of $X^{(1)}_I(\zeta'_0)$ are independent of the singular [illegible] $\operatorname{Ker}\tilde{A}$ of the $\kappa$-matrix $\tilde{L}(\kappa,\zeta_0)$. Obviously each one of the columns $X_k(\zeta'_0) = X_k(\zeta_0)$ is independent of $\operatorname{Ker}\tilde{A}$, and according to part c) of lemma 8.2 the columns of $X^{(2)}_\infty(\zeta'_0)$ are also independent of the above eigenspace. Then the above Jordan sequence is regular and hence, according to lemma 7.2, the vectors of the sequence are independent of $\operatorname{Ker}\tilde{A}$. Hence the columns of $X^{(2)}(\zeta'_0) = (X^{(2)}_F(\zeta'_0), X^{(2)}_\infty(\zeta'_0))$ are independent of the space $\operatorname{Sp} X^{(1)}_I(\zeta'_0) + \operatorname{Ker}\tilde{A} = \operatorname{Ker}\tilde{P}(0)$. Then part [illegible] of the lemma follows from continuity of $X^{(2)}(\zeta')$ and $\tilde{P}(\xi)$ as functions of $\zeta'$.

Lemma 8.7 a) The matrix $\hat{T}^{-1}(\zeta') = rz'T^{-1}(\zeta')$ and also the matrices $z'(T^{-1}(\zeta'))^{(1)}_{F1}$ and $(T^{-1}(\zeta'))^{(2)}$ are analytic in $\Omega(\zeta'_0)$.

b) The last row of the matrix $(\hat{T}^{-1}(\zeta'_0))^{(1)}_{\infty}$ is non-zero.

Proof: Since the matrix $X(\zeta'_0)$ is singular, the proof used in lemma 8.5 for the analyticity of $\hat{T}^{-1}(\zeta')$ is now unacceptable. Let us integrate the matrix $X(\zeta')(\tilde{B}_0(\zeta')+\kappa\tilde{B}_1(\zeta'))^{-1}T^{-1}(\zeta')$ for $\zeta'\in\Omega_R(\zeta'_0)$ around the unit circle $|\kappa| = 1$. Since the integral $\int_{|\kappa|=1} (\tilde{B}_0(\zeta') + \kappa\tilde{B}_1(\zeta'))^{-1}d\kappa = I_n \oplus 0_{(m-1)n}$, where the unit matrix corresponds to the blocks $M_0$ and $M_I$ and the matrix $0_{(m-1)n}$ to $M_{II}$ and $M_{\infty}$, we get from (8.23) an estimate

$$\| (X_0,X_I)\cdot[(T^{-1})_0,(T^{-1})_I]' \| \leq \frac{K}{|z|-1}$$

where the variable $\zeta'$ is omitted. The independence of the columns of $(X_0,X_I)$ implies that

$$\| (T^{-1})_0 \| \ , \ \| (T^{-1})_I \| \leq \frac{K}{|z|-1} .$$

Let us fix in (8.23) a value of $\kappa$ bounded away from $\kappa = 1$ and with $|\kappa| = 1$. Then we get also

$$\| (X_{II},X_{\infty})[(\kappa I-M_{II})^{-1}\oplus(I-\kappa M_{\infty})^{-1}][(T^{-1})_{II},(T^{-1})_{\infty}] \| \leq \frac{K}{|z|-1}$$

and finally

$$\| T^{-1}(\zeta') \| \leq \frac{K}{|z|-1} . \tag{8.24}$$

Let us note that the matrix $T(\zeta')$ is non-singular for $\zeta'\in\Omega(\zeta'_0)$, when $z'r \neq 0$. Therefore the zeros of the function $|T(\zeta')|$ are also zeros of the function $z'r$, and using, for example, the Nullstellensatz (see [9]) one can show that $|T(\zeta')| = (z')^{k_1}r^{k_2}\varphi(\zeta')$, where the function $\varphi(\zeta')$ is analytic in $\Omega(\zeta'_0)$ and $\varphi(\zeta'_0) \neq 0$. Then the matrix-function $\hat{T}^{-1}(\zeta')$ has singularity of the type $(z')^{-k_1}r^{-k_2}$. Let us take $\zeta' = (\xi',z',r) \in \Omega_R(\zeta'_0)$ with Re $z' > 0$ and fix

$\xi'$ and $z'$. Representing an arbitrary element of $\hat{T}^{-1}(\zeta')$ as a fraction $\psi(\zeta')/((z')^{k_1} r^{k_2})$, where $\psi(\zeta')$ is analytic in $\Omega(\zeta_0')$, we obtain from (8.24) an estimate

$$|\psi(\zeta')/((z')^{k_1} r^{k_2})| \le K\cdot|rz'|/(|z|-1) \le K . \qquad (8.25)$$

Therefore $\psi(\zeta')$ may be reduced by $r^{k_2}$ for the above $\zeta'$ and, therefore, for any $\zeta'\in\Omega(\zeta_0')$. Similarly, let us fix in $\zeta'\in\Omega_R(\zeta_0')$ the components $\xi'$ and $r$ and let the variable $z'$ be real and positive. Then from (8.25) follows that $\psi(\zeta')$ may be reduced by $(z')^{k_2}$ for any $\zeta'\in\Omega(\zeta_0)$ and therefore the matrix $\hat{T}^{-1}(\zeta')$ is analytic in $\Omega(\zeta_0')$. Let us now prove that $(\hat{T}^{-1}(\zeta')_{F1}^{(1)} = 0$ for $r = 0$ and $(\hat{T}^{-1}(\zeta'))^{(2)} = 0$ if $rz' = 0$. The equality $T(\zeta')\hat{T}^{-1}(\zeta') = rz'I$ implies that $\operatorname{Im} \hat{T}^{-1}(\zeta') \subset \operatorname{Ker} T(\zeta')$ for $rz' = 0$. Let $z' \neq 0$, $r = 0$ and, therefore, $\zeta = \zeta_0$. According to part a) of lemma 8.6 the columns of $(X_{F1}^{(1)}(\zeta'),X^{(2)}(\zeta'))$ are independent of the columns of $(X_0(\zeta_0),\ X_\infty^{(1)}(\zeta_0))$, which form the basis of Ker $\tilde{A}$. Taking $v\in\operatorname{Ker} T(\zeta')$ and proceeding as in the proof of part b) of lemma 8.5 we derive that $v_{F1}^{(1)} = v^{(2)} = 0$ and therefore $(\hat{T}^{-1}(\zeta'))_{F1}^{(1)} = (\hat{T}^{-1}(\zeta'))^{(2)} = 0$. Let now $z' = 0$ and $r \neq 0$. According to lemma 8.6 the columns of $X^{(2)}(\zeta')$ are independent of the singular eigenspace $\operatorname{Ker} \tilde{P}(\xi) = \tilde{V}_0(\xi)$ of the singular $\kappa$-matrix $\tilde{L}(\kappa,\zeta)$, where $\zeta = (\xi'\cdot r,1)$. Taking $v\in\operatorname{Ker} T(\zeta')$ and following the analyticity proof of $(T^{-1}(\zeta))^{(2)}$ in lemma 7.6, we conclude that $v^{(2)} = 0$ and therefore $(\hat{T}^{-1}(\zeta'))^{(2)} = 0$. The matrix $(\hat{T}^{-1}(\zeta'))^{(2)}$ is therefore divisible by $rz'$ and $(\hat{T}^{-1}(\zeta'))_{F1}^{(1)}$ by $r$, so that part a) of the lemma is proved.

In order to prove part b) of our lemma we shall construct a vector function $v(\zeta')$ analytic in $\Omega(\zeta_0')$ such that the last component of

$v_\infty^{(1)}(\zeta'_0)$ is non-zero and $T(\zeta')v(\zeta') = O(rz')$. Then multiplying the last equality on the left by $T^{-1}(\zeta')$ we obtain that $v(\zeta'_0) \in \operatorname{Im} \hat{T}^{-1}(\zeta'_0)$. Let us fix $\kappa$ different from all the eigenvalues of $\tilde{B}_0(\zeta') + \kappa\tilde{B}_1(\zeta')$ for any $\zeta' \in \Omega(\zeta'_0)$. The vector $\tilde{\varphi}_0(\kappa,0) \in \operatorname{Ker} \tilde{A}$ may be represented as a linear combination

$$\tilde{\varphi}_0(\kappa,0) = X_0(\zeta'_0)u_0(\zeta'_0) + X_\infty^{(1)}(\zeta'_0)u_\infty^{(1)}(\zeta'_0).$$

As in the proof of part b) of lemma 8.5, we may assume that the last component of $u_\infty^{(1)}(\zeta'_0)$ is non-zero. Let us define a vector $u(\zeta'_0) \in C^{mn}$ by adding to $u_0(\zeta'_0)$ and $u_\infty^{(1)}(\zeta'_0)$ zeros in the remaining components. Then for $\zeta' = (\xi',z',r) \in \Omega(\zeta'_0)$ and the corresponding $\zeta = (\xi,z)$, we arrive at

$$\tilde{\varphi}_0(\kappa,\xi) - X(\zeta')u(\zeta'_0) = r\Delta\varphi(\zeta'),$$

where $\Delta\varphi(\zeta')$ is analytic in $\Omega(\zeta'_0)$ and for $z' = 0$, $\Delta\varphi(\zeta') \in \operatorname{Ker} \tilde{P}(\xi)$. Since the columns of $(X_0(\zeta'), X_I^{(1)}(\zeta'), X_\infty^{(1)}(\zeta'))$ form for $\zeta' = (\xi',0,r)$ a basis of $\operatorname{Ker} \tilde{P}(\xi)$, there exists a vector function $\Delta u(\zeta')$ analytic in $\Omega(\zeta'_0)$ such that

$$\Delta u^{(2)}(\zeta') = 0 \quad \text{and} \quad \Delta\varphi(\zeta') - X^{(1)}(\zeta')\Delta u^{(1)}(\zeta') = O(z').$$

Therefore defining $u(\zeta') = u(\zeta'_0) + r\Delta u(\zeta')$ we obtain

$$\tilde{\varphi}_0(\kappa,\xi) - X(\zeta')u(\zeta') = O(rz') \ .$$

Since $\tilde{L}(\kappa,\zeta)\tilde{\varphi}_0(\kappa,\xi) = O(z-1) = O(rz')$, we obtain for the vector-function $v(\zeta') = (\tilde{B}_0(\zeta')+\kappa\tilde{B}_1(\zeta'))u(\zeta')$ an estimate

$$T(\zeta')v(\zeta') = \tilde{L}(\kappa,\zeta)X(\zeta')u(\zeta') = O(rz').$$

To accomplish the proof we should note that the matrix $I-\kappa M_\infty^{(1)}(\zeta_0')$ is of upper triangular form with the unit main diagonal. Since $v_\infty^{(1)}(\zeta_0') = (I-\kappa M_\infty^{(1)}(\zeta'))\cdot u_\infty^{(1)}(\zeta_0')$, the last component of $v_\infty^{(1)}(\zeta_0')$ is equal to the last one of $u_\infty^{(1)}(\zeta_0')$ and, thus, it is non-zero. Q.E.D.

## 8.2. Construction of the Kreiss symmetrizer for the matrix $M_j^{(1)}(\zeta') = I+rM_j'(\zeta')$ in the case $\mathrm{Re}\kappa_j' = 0$.

Let $\mathrm{Re}\ z_0' = 0$, $z_0' \neq 0$ and suppose that the matrix $M_j'(\zeta_0')$ has the eigenvalue $\kappa_j'$ with $\mathrm{Re}\ \kappa_j' = 0$. According to lemma 8.3 we may assume that $M_j'(\zeta_0')$ is a Jordan cell of the order $q_j^{(1)}$. For ease of notation we shall write $q$ instead of $q_j^{(1)}$. Following Gustafsson et al [3] we consider a matrix

$$\hat{M}_j'(\zeta') = -(i/r)\ell n\ M_j^{(1)}(\zeta') = -(i/r)\ell n(I + rM_j'(\zeta')) . \tag{8.26}$$

Obviously the matrix $\hat{M}_j'(\zeta')$ is analytic in $\Omega(\zeta_0')$ and $\hat{M}_j'(\zeta_0') = -iM_j'(\zeta_0')$ is a Jordan cell with the real eigenvalue $\hat{\kappa}_j' = -i\kappa_j'$. The matrix $X_j^{(1)}(\zeta')$ may be chosen in such a way that $\hat{M}_j'(\zeta')$ has a form

$$\hat{M}_j'(\zeta') = \hat{\kappa}_j'\cdot I + \begin{pmatrix} e_{q-1} & 1 & 0 & . & . & 0 \\ e_{q-2} & 0 & 1 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & 1 \\ e_0 & 0 & . & . & . & 0 \end{pmatrix} \tag{8.27}$$

where $e_k = e_k(\zeta')$, $k = 0,1,\ldots,q-1$, depend analytically $\zeta'$ and vanish at the point $\zeta' = \zeta_0'$. Let $\zeta' \in \Omega_R(\zeta_0')$. Denote by $\rho$ the number of the eigenvalues $\kappa$ of the matrix $M_j^{(1)}(\zeta')$ with $|\kappa| < 1$. Since the $\kappa$-matrix $L(\kappa,\zeta')$ has no eigenvalues with $|\kappa| = 1$ for $\zeta' \in \Omega_R(\zeta_0')$, it follows that the number $\rho$ is independent of $\zeta'$. It is easy to show that the mapping $\kappa \to \hat{\kappa}' = -(i/r)\, \ell n\, \kappa$ transforms the eigenvalues of $M_j^{(1)}(\zeta')$ into the eigenvalues of $\hat{M}_j'(\zeta')$ so that for $\kappa$ with $|\kappa| < 1$ we have $\text{Im}\, \kappa' > 0$ and vice versa. Thus, the matrix $\hat{M}_j'(\zeta')$ has $\rho$ eigenvalues in the halfplane $\text{Im}\, \hat{\kappa}' > 0$ and $q-\rho$ in the half plane $\text{Im}\, \kappa' < 0$. Let us partition the matrix $X_j^{(1)}(\zeta')$ as

$$X_j^{(1)} = (X_{I,j}^{(1)}, X_{II,j}^{(1)}) \tag{8.28}$$

where the matrix $X_{I,j}^{(1)}$ consists of the first $\rho$ columns of $X_j^{(1)}$ and $X_{II,j}^{(1)}$ of the remaining $q-\rho$ ones. If $v_j^{(1)}$ is a q-dimensional column-vector, we shall similarly partition it as

$$v_j^{(1)} = (v_{I,j}^{(1)}, v_{II,j}^{(1)})' , \tag{8.29}$$

where ' is now the transposition symbol. As in (3.16) we have a matrix $U_j^{(1)}(\zeta')$ continuous at the point $\zeta_0'$ such that $U_j^{(1)}(\zeta_0') = I$ and

$$(U_j^{(1)}(\zeta'))^{-1} M_j^{(1)}(\zeta') U_j^{(1)}(\zeta') = \begin{pmatrix} \kappa_{j1} & \gamma & 0 & . & . & 0 \\ 0 & \kappa_{j2} & \gamma & . & . & 0 \\ . & & & & & . \\ . & & & & & . \\ . & & & & & \gamma \\ 0 & & & & & \kappa_{jq} \end{pmatrix} = \begin{pmatrix} N_{j11} & N_{j12} \\ 0 & N_{j22} \end{pmatrix} . \tag{8.30}$$

Here $\gamma = ir$, and for $\zeta' \in \Omega_R(\zeta_0')$ the first $\rho$ eigenvalues $\kappa_{j1}, \kappa_{j2}, \ldots, \kappa_{j\rho}$ satisfy $|\kappa_{jk}| < 1$ and remaining $q-\rho$ ones have $|\kappa_{jk}| > 1$, so that the spectra of the matrices $N_{j11}$ and $N_{j22}$ lie correspondingly inside and outside the unit circle $|\kappa| = 1$.

The main result of this subsection is

Theorem 8.1. There exists a Hermitian matrix $R_j^{(1)}(\zeta')$ depending smoothly on $\zeta' \in \Omega(\zeta_0')$ and satisfying

$$(8.31)\quad (v_j^{(1)})^* R_j^{(1)}(\zeta') v_j^{(1)} \geq -c|v_{I,j}^{(1)}|^2 + |v_{II,j}^{(1)}|^2 \quad \text{for any } \zeta' \in \Omega(\zeta_0'),$$

$$(8.32)\quad (M_j^{(1)}(\zeta'))^* R_j^{(1)}(\zeta') M_j^{(1)}(\zeta') - R_j^{(1)}(\zeta') \geq \delta(|z|-1)I \quad \text{for any } \zeta' \in \Omega_R(\zeta_0')$$

where $\delta$ and $c$ are positive constants and $c$ may be chosen arbitrarily small.

We shall use the methods of Kreiss in [2] in order to construct the above symmetrizer for the matrix $i\hat{M}_j'(\zeta')$, so that in addition to (8.31) the estimate

$$(8.33)\qquad \operatorname{Re}(iR_j^{(1)}(\zeta')\hat{M}_j'(\zeta')) \geq \delta\left(\frac{|z|-1}{r}\right) I$$

holds for any $\zeta' \in \Omega_R(\zeta_0')$. Then as in [3] one obtains for the matrix $M_j^{(1)}(\zeta') = \exp(ir\hat{M}_j'(\zeta'))$ the estimate (8.32). Unfortunately the coefficients $e_k(\zeta')$ in (8.27) do not satisfy the condition of the Ralston's note [8]. For example, $e_k(\zeta')$ are not real for $|z| = 1$. The following lemma provides, however, the necessary estimates for the imaginary part of $e_k(\zeta')$.

Lemma 8.8. There is a neighbourhood $\Omega_R(\zeta_0')$ and positive constants $K$ and $\delta$ such that the estimates

$$(8.34)\qquad |\operatorname{Im} e_k(\zeta')| \leq K|\operatorname{Im} e_0(\zeta')|, \quad k = 1,\dots,q-1$$

$$(8.35)\qquad |\operatorname{Im} e_0(\zeta')| \geq \delta\left(\frac{|z|-1}{r} + r^3\right) \geq \delta\frac{|z|-1}{r}$$

hold for any $\zeta' \in \Omega_R(\zeta_0')$.

Proof: We shall not take advantage of the specific form of our difference approximation. What will be essential in our proof is the dissipativity of the difference scheme.

For any complex r consider a mapping

$$\kappa' = \varphi(\hat{\kappa}',r) = (\exp(i\hat{\kappa}'r)-1)/r \; . \tag{8.36}$$

The function $\varphi(\hat{\kappa}',r)$ depends analytically on $\hat{\kappa}'$ and $r$ (including $r = 0$) and the mapping $\hat{\kappa}' \to \varphi(\hat{\kappa}',r)$ is one-to-one for bounded $\hat{\kappa}'$ and sufficiently small r. Since $M_j'(\zeta') = \varphi(\hat{M}_j'(\zeta'),r)$, the mapping in (8.36) transforms the roots of the equation $|\hat{M}_j'(\zeta')-\hat{\kappa}'I| = 0$ into the roots of the equation $|M_j'(\zeta')-\kappa'I| = 0$. Denote $\hat{L}'(\hat{\kappa}',\zeta') = L'(\varphi(\hat{\kappa}',r),\zeta')$. Then the mapping in (8.36) provides a one-to-one correspondence between the roots of the equations $|\hat{L}'(\hat{\kappa}',\zeta')| = 0$ and $|L'(\kappa',\zeta')| = 0$. Since the equations $|M_j'(\zeta')-\kappa'I| = 0$ and $|L'(\kappa',\zeta')| = 0$ are equivalent in $\Omega(\kappa_j')$, it follows that the equations $|\hat{M}_j'(\zeta')-\hat{\kappa}'I| = 0$ and $|\hat{L}'(\hat{\kappa}',\zeta')| = 0$ are equivalent in a neighbourhood $\Omega(\hat{\kappa}_j')$ of the point $\hat{\kappa}_j'$. The matrix $\hat{L}'(\hat{\kappa}',\zeta')$ is connected with the amplification matrix G in (5.23) and is given by

$$\hat{L}'(\hat{\kappa}',\zeta') = \exp(i\hat{\kappa}'r)(z'I-G'(\hat{\kappa}',\xi',r))$$

where

$$G'(\hat{\kappa}',\xi',r) = (G(\hat{\kappa}'\cdot r,\xi'\cdot r)-I)/r$$

(the factor $\exp(i\hat{\kappa}'r) = \kappa$ is due to the fact that the original difference operator L in (5.2) was multiplied later in (5.21) by the shift operator $E_x$). The consistency of the difference approximation implies that

$$G'(\hat{\kappa}',\xi',0) = -i(A\hat{\kappa}'+B\xi') \; .$$

Since $\hat{\kappa}'_j$ is real and $|\hat{\kappa}'_j| + |\xi'_0| \neq 0$, the matrix $A\hat{\kappa}'_j + B\xi'_0$ has distinct eigenvalues and therefore the matrix $G'(\hat{\kappa}',\xi',r)$ is diagonalizable for any $(\hat{\kappa}',\zeta') \in \Omega(\hat{\kappa}'_j)\times\Omega(\zeta'_0)$:

$$G'(\hat{\kappa}',\xi',r) \sim \mathrm{diag}(g'_1,g'_2,\dots,g'_n),$$

where $g'_k = g'_k(\hat{\kappa}',\xi',r)$, $k = 1,2,\dots,n$, depends analytically on $\hat{\kappa}'$, $\xi'$ and $r$. In our characteristic case we may assume that $g'_1(\hat{\kappa}',\xi',0) = 0$ and therefore $g'_1(\hat{\kappa}',\xi',r) = O(r)$. Since $z'_0 \neq 0$, the equation $|\hat{L}'(\hat{\kappa}',\zeta')| = 0$ for $\zeta' \in \Omega(\zeta'_0)$ is equivalent to the following $n$-1 equations

(8.37) $$z'-g'_k(\hat{\kappa}',\xi',r) = 0 \ , \quad k = 2,3,\dots,n \ .$$

Since the values $g'_k(\hat{\kappa}'_j,\xi'_0,0)$, $2 \leq k \leq n$, are distinct, it follows that $\hat{\kappa}' = \hat{\kappa}'_j$ is a root of only one equation of the type (8.37), namely for such k, $2 \leq k \leq n$, which satisfies $g'_k(\hat{\kappa}'_j,\xi'_0,0) = z'_0$. We shall omit the index k in this specific function

$$g'(\hat{\kappa},\xi',r) = g'_k(\hat{\kappa},\xi',r)$$

and rewrite the corresponding equation (8.37) as

$$z' - g'(\hat{\kappa},\xi',r) = 0 \ .$$

Let us denote by $g(\hat{\kappa},\xi',r) = 1+rg'(\hat{\kappa},\xi',r)$ the corresponding eigenvalue of the amplification matrix $G(\hat{\kappa}'\cdot r,\xi'\cdot r)$. Then the last equation may be written in the following equivalent form

(8.38) $$f(\hat{\kappa}',\zeta') = \frac{\ln z}{ir} - \frac{\ln g(\hat{\kappa},\xi',r)}{ir} = 0 \quad (\text{where } z = 1+rz') \ .$$

The function $f(\hat{\kappa}',\zeta')$ is analytic in $\Omega(\hat{\kappa}_j')\times\Omega(\zeta_0')$. For $\zeta'\in\Omega(\zeta_0')$ the characteristic equation $|\hat{M}_j'(\zeta')-\hat{\kappa}'I| = 0$ is equivalent in $\Omega(\hat{\kappa}_j')$ to equation (8.38). Since

$$|\hat{\kappa}'I-\hat{M}_j'(\zeta')| = (\hat{\kappa}'-\hat{\kappa}_j')^q-e_{q-1}(\zeta')\cdot(\hat{\kappa}'-\hat{\kappa}_j')^{q-1}-\ldots-e_0(\zeta'),$$

it follows that $\pm e_k(\zeta')$, $k=0,1,\ldots,q-1$, are coefficients of the Weierstrass polynomial corresponding to the function $f(\hat{\kappa}',\zeta')$. Define a function

$$\bar{f}(\hat{\kappa}',\zeta') = \overline{f(\bar{\hat{\kappa}}',\zeta')},$$

where — is a symbol of complex conjugation. The function $\bar{f}(\hat{\kappa}',\zeta')$ is analytic in $\hat{\kappa}'$ but not in $\zeta'$. For $\zeta'\in\Omega_R(\zeta_0')$ we have

$$f(\hat{\kappa}',\zeta')-\bar{f}(\hat{\kappa}',\zeta') = [\ell n(|z|^2)-\ell n(g(\hat{\kappa}',\xi',r)\cdot\overline{g(\bar{\hat{\kappa}}',\xi',r)})]/(ir)\ . \tag{8.39}$$

According to estimate (5.27) our difference scheme is dissipative of order 4. Therefore for real $\hat{\kappa}',\xi'$ and $r$ there is an estimate

$$g(\hat{\kappa}',\xi',r)\cdot\overline{g(\bar{\hat{\kappa}}',\xi',r)} = |g(\hat{\kappa}',\xi',r)|^2 \le 1-\delta r^4 \tag{8.40}$$

provided $|\hat{\kappa}'| + |\xi'|$ is bounded away from zero. Consider an analytic function of the complex variables $\hat{\kappa}',\xi'$ and $r$

$$h(\hat{\kappa}',\xi',r) = g(\hat{\kappa}',\xi',r)\cdot\overline{g(\bar{\hat{\kappa}}',\bar{\xi}',\bar{r})}$$

and let us expand it in a power series according to $r$

$$h(\hat{\kappa}',\xi',r) = 1 + \sum_{i=1}^{\infty} h_i(\hat{\kappa}',\xi')r^i\ .$$

Let $\hat{\kappa}',\xi'$ and r be real. Then $h(\hat{\kappa}',\xi',r) = |g(\hat{\kappa}',\xi',r)|^2$ and it follows from (8.40) that the first non-zero coefficient $h_i(\hat{\kappa}',\xi')$ should have an even index $i = 2m \le 4$ and should be negative. Actually $m = 2$, since otherwise the scheme would be dissipative of order less than 4. Therefore

$$h(\hat{\kappa}',\xi',r) = 1 + O(r^4)$$

also for complex $\hat{\kappa}',\xi'$ and r. Let now $\hat{\kappa}' \in \Omega(\hat{\kappa}'_j)$ be complex and $\zeta' \in \Omega_R(\zeta'_0)$, i.e. $\xi'$ and r are real. Then

$$g(\hat{\kappa}',\xi',r)\cdot\overline{g(\bar{\hat{\kappa}}',\xi',r)} = h(\hat{\kappa}',\xi',r) = 1+O(r^4) \tag{8.41}$$

and

$$\frac{\partial}{\partial\hat{\kappa}'}\left(g(\hat{\kappa}',\xi',r)\cdot\overline{g(\bar{\hat{\kappa}}',\xi',r)}\right) = O(r^4)\,. \tag{8.42}$$

It follows now from (8.39), (8.41) and (8.42) that

$$|f(\hat{\kappa}',\zeta') - \bar{f}(\hat{\kappa}',\zeta')| \le K\left(\frac{|z|-1}{r} + r^3\right) \tag{8.43}$$

and

$$\left|\frac{\partial f(\hat{\kappa}',\zeta')}{\partial\hat{\kappa}'} - \frac{\partial \bar{f}(\hat{\kappa}',\zeta')}{\partial\hat{\kappa}'}\right| \le Kr^3\,. \tag{8.44}$$

Denote $f_0(\zeta') = f(\hat{\kappa}'_j,\zeta')$. Then $\mathrm{Im}\, f_0(\zeta') = (f(\hat{\kappa}'_j,\zeta')-\bar{f}(\hat{\kappa}'_j,\zeta'))/(2i)$ and using (8.39) and estimate (8.40) for real $\hat{\kappa}' = \hat{\kappa}'_j$ we obtain

$$|\mathrm{Im}\, f_0(\zeta')| \ge \delta\left(\frac{|z|-1}{r} + r^3\right)\,. \tag{8.45}$$

Therefore one can replace the right hand sides of estimates (8.43) and (8.44) by $K|\mathrm{Im}\, f_0(\zeta')|$. Thus, the function $f(\hat{\kappa}',\zeta')$ satisfies the conditions of lemma 8.9 proven below. The correspondence with the notations of the lemma is as follows:

$$z_1 = \hat{\kappa}'-\hat{\kappa}'_j\,, \quad w = \zeta'-\zeta'_0 \text{ and } D = \Omega(\zeta'_0)-\zeta'_0\,.$$

Since the functions $\pm e_k(\zeta')$, $k = 0,1,\dots,q-1$, are coefficients of the Weierstrass polynomial corresponding to $f(\hat{\kappa}',\zeta')$, we arrive according to lemma 8.9 and estimate (8.45) at the required estimates (8.34) and (8.35). Our proof is valid for any dissipative difference approximation. If the order of dissipativity is m instead of 4, one should replace $r^3$ in (8.35) by $r^{m-1}$.

It follows from (8.35) that $\operatorname{Im} e_0(\zeta')$ is of constant sign for $\zeta' \in \Omega_R(\zeta'_0)$. As in [2] (lemma 2.7) we shall show that the number $\rho$ of the eigenvalues of $\hat{M}'_j(\zeta')$ in the halfplane $\operatorname{Im} \hat{\kappa}' > 0$ is given by formula (3.14). Since this number $\rho$ is independent of $\zeta' \in \Omega_R(\zeta'_0)$, we shall take $\zeta' = (\xi'_0, z', 0)$ with $\operatorname{Im} z' = \operatorname{Im} z'_0$ and $\operatorname{Re} z' > 0$. Such point does not belong to $\Omega_R(\zeta'_0)$, but is a limit point of a set $\{(\xi'_0, z', r) \in \Omega_R(\zeta'_0), r > 0\}$. Then $|\operatorname{Im} e_0(\zeta')| \geqslant \delta \operatorname{Re} z'$ and $e_k(\zeta') = O(|\zeta' - \zeta'_0|) = O(\operatorname{Re} z')$, $k = 0,1,\dots,q-1$. Therefore the eigenvalues $\hat{\kappa}'$ of the matrix $\hat{M}'_j(\zeta')$ may be written in a form

$$\hat{\kappa}' = \hat{\kappa}'_j + (e_0(\zeta'))^{1/q} \cdot (1 + O(\operatorname{Re} z')^{1/q})$$

and formula (3.14) follows easily.

Using estimates (8.34) and (8.35) and formula (3.14) we are able to construct the required symmetrizer $R_j^{(1)}(\zeta')$ for the matrix $\hat{M}'_j(\zeta')$. Using the notations of Kreiss in [2], the matrix $i\hat{M}'_j(\zeta')$ is represented as

$$i\hat{M}'_j(\zeta') = i\hat{\kappa}'_j \cdot I + iC + iE(\zeta') + N(\zeta'), \text{ where}$$

$$C = \begin{pmatrix} 0 & 1 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \end{pmatrix}, \quad E(\zeta') = \begin{pmatrix} \operatorname{Re} e_{q-1} & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \operatorname{Re} e_0 & & & \cdot & \cdot & 0 \end{pmatrix},$$

$$N'(\zeta') = -\begin{pmatrix} \operatorname{Im} e_{q-1} & \cdots & \cdots \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \operatorname{Im} e_{0} & \cdots & \cdots \end{pmatrix}$$

Then $R_j^{(1)}(\zeta') = (D+B)-iF$, where D, B and F are corresponding the matrices D, $\varepsilon B$ and $\eta' F$ defined in [?] lemmas [illegible].

So theorem 8.1 is proved modulo the following general

Lemma 8.9. Let $f(z_1,w)$, where $w = (z_2,z_3,\dots,z_n)$ be a function of n complex variables analytic in a neighbourhood of the point $z_1 = 0$, $w = 0$, such that $f(0,0) = 0$ and $f(z_1,0)$ is regular of order q in $z_1$. Denote by

$$P(z_1,w) = \sum_{k=0}^{q-1} e_k(w) z_1^k + z_1^q$$

the Weierstrass polynomial corresponding to the function $f(z_1,w)$. Suppose that for any w belonging to some set $M \subset \mathbb{C}^{n-1}$ and any $z_1$ in a neighbourhood of zero the following estimate holds:

$$(8.46) \qquad |\bar{f}(z_1,w) - f(z_1,w)| \le K|\operatorname{Im} f_0(w)|$$

$$(8.47) \qquad \left|\frac{\partial \bar{f}}{\partial z_1}(z_1,w) - \frac{\partial f}{\partial z_1}(z_1,w)\right| \le K|\operatorname{Im} f_0(w)| ,$$

where $\bar{f}(z_1,w) = \overline{f(\bar{z}_1,w)}$ and $f_0(w) = f(0,w)$.

Then there is some neighbourhood $\Omega_w \subset \mathbb{C}^{n-1}$ of the point $w = 0$ and positive constants $K_1$ and $\delta$ such that for any $w \in M \cap \Omega_w$ the estimates

$$(8.48) \qquad |\operatorname{Im} e_k(w)| \le K_1 |\operatorname{Im} f_0(w)| , \quad k = 0,1,\dots,q-1,$$

and

(5.49) $$|\mathrm{Im}\, e_0(w)| \geq \delta |\mathrm{Im}\, f_0(w)|$$

hold.

Proof: Let $\Delta = \Delta_{z_1} \times \Delta_w$ be a polydisk in a neighbourhood of the point $z_1 = 0$, $w = 0$, where $\Delta_{z_1} = \{z_1 \in \mathbb{C} \,|\, |z_1| \leq \varepsilon_1\}$, $\Delta_w = \{w \in \mathbb{C}^{n-1} \,|\, |w| \leq \varepsilon_2\}$, such that the function f is analytic in $\Delta$. Define a contour $\Gamma = \{z_1 \in \mathbb{C} \,|\, |z_1| = \varepsilon_3\}$ with $\varepsilon_3 < \varepsilon_1$ and polydisk $\Delta'_w \subset \Delta_w$ such that for any $(z_1, w) \in \Gamma \times \Delta'_w$, $f(z_1, w) \neq 0$. Let us fix $w \in \Omega \cap \Delta'_w$. Denote by $z_{11}, z_{12}, \ldots, z_{1q}$ the roots of the equation $f(z_1, w) = 0$.

Then

$$\sigma_j(z_{11}, z_{12}, \ldots, z_{1q}) = z_{11}^j + z_{12}^j + \ldots + z_{1q}^j = \frac{1}{2\pi i} \int_\Gamma \frac{z_1^j f'(z_1, w)}{f(z_1, w)} dz_1 ,$$

where $f'(z_1, w) = \dfrac{\partial f(z_1, w)}{\partial z_1}$. Therefore

$$\bar{\sigma}_j = \bar{z}_{11}^j + \bar{z}_{12}^j + \ldots + \bar{z}_{1q}^j = \frac{1}{2\pi i} \oint_\Gamma \frac{z_1^j \bar{f}'(z_1, w)}{\bar{f}(z_1, w)} dz_1 ,$$

$$\mathrm{Im}\,\sigma_j = -\frac{1}{4\pi} \oint_\Gamma z_1^j \left(\frac{f'}{f} - \frac{\bar{f}'}{\bar{f}}\right) dz_1 = -\frac{1}{4\pi} \oint_\Gamma z_1^j \frac{f'(\bar{f} - f) + f(f' - \bar{f}')}{f \cdot \bar{f}} dz_1 .$$

Since $f(z_1, w) \cdot \bar{f}(z_1, w) \neq 0$ for any $(z_1, w) \in \Gamma \times \Delta'_w$, we get from (5.48) and (5.47) the estimate

(5.50) $$|\mathrm{Im}\,\sigma_j(w)| \leq K |\mathrm{Im}\, f_0(w)| , \quad j = 1, \ldots, q ,$$

where the constant K is independent of $w \in \Omega \cap \Delta'_w$. Since $e_k(w)$ is a polynomial in $\sigma_1, \ldots, \sigma_q$, with real coefficients, we obtain for $\mathrm{Im}\, e_k(w)$ the esti- [illegible]

$$e_j(w) = [illegible] \frac{(-1)^{j-1}}{j} \sigma_j(w) + [illegible](\sigma_1, \ldots, \sigma_{j-1}) ,$$

[illegible] from the considera- [illegible]

$\sigma_1, \sigma_2, \ldots, \sigma_{q-1}$. Since $\sigma_j(0) = 0$ it follows that $|\operatorname{Im} s(\sigma_1, \sigma_2, \ldots, \sigma_{q-1})| \leq \delta_1 |\operatorname{Im} f_0(w)|$, where $\delta_1$ is arbitrarily small if one sets $\varepsilon_2$ small enough. Let us write the integral expression for $\operatorname{Im} \sigma_q(w)$ :

$$\operatorname{Im} \sigma_q(w) = -\frac{1}{4\pi}\left(\oint_\Gamma z_1^q \frac{f'}{f\cdot\bar{f}} \cdot (\bar{f}-f)dz_1 + \oint_\Gamma \frac{z_1^q}{\bar{f}} \cdot (f'-\bar{f}')dz_1\right).$$

The functions $\dfrac{z_1^q f'(z_1,w)}{f(z_1,w)\bar{f}(z_1,w)}$ and $\dfrac{z_1^q}{\bar{f}(z_1,w)}$ are continuous on $\Gamma \times \Delta'_w$ and, hence, the difference of their values at the points $(z_1,w)$ and $(z_1,0)$ may be bounded by arbitrarily small constant if one chooses $\varepsilon_2$ in a corresponding way. Let us note that $\dfrac{z_1^q}{\bar{f}(z_1,0)}$ is analytic function of $z_1$ and therefore

$$\oint_\Gamma \frac{z_1^q}{\bar{f}(z_1,0)} (f'(z_1,w) - \bar{f}'(z_1,w))dz_1 = 0 .$$

Similarly,

$$\frac{z_1^q \cdot f'(z_1,0)}{f(z_1,0)\cdot \bar{f}(z_1,0)} = \frac{q}{\bar{f}_q(0,0)} \cdot \frac{1}{z_1} + g(z_1),$$

where $\bar{f}_q(0,0) = \left.\dfrac{\partial^q \bar{f}(z_1,0)}{q!\,\partial z_1^q}\right|_{z_1=0}$ is non-zero and $q(z_1)$ is analytic function.

Then

$$-\frac{1}{4\pi} \oint_\Gamma \frac{z_1^q \cdot f'(z_1,0)}{f(z_1,0)\bar{f}(z_1,0)} (\bar{f}(z_1,w)-f(z_1,w))dz = \frac{-q\cdot \operatorname{Im} f_0(w)}{\bar{f}_q(0,0)} .$$

Now, using estimates (8.46) and (8.47), we obtain for sufficiently small $\varepsilon_2$

$$|\operatorname{Im} \sigma_q(w)| \leq \delta |\operatorname{Im} f_0(w)|$$

and therefore

$$|\mathrm{Im}\, e_0(w)| \le \delta |\mathrm{Im}\, f_0(w)|$$

for some constant $\delta$ independent of $w \in D \cap \Delta_w'$.

## 8.3. Proof of theorems 5.1-5.3 in the neighbourhood $\Omega(\zeta_0')$.

We consider first the case $z_0' \neq 0$. The operator P in estimate (6.9) is now defined as $P = \tilde{A} = \mathrm{diag}(A, A, \ldots, A)$. Theorem 5.3 is formulated now in a following form:

Sufficiency: If (UKC) is satisfied in $\Omega(\zeta_0')$ and $\dim \tilde{S}(0,1)\mathrm{Ker}\,\tilde{A} = 1$, estimate (6.9) holds in $\Omega(\zeta_0')$ with $|z_0| = 1$.

Necessity: If estimate (6.9) holds in $\Omega(\zeta_0')$ with $|z_0| = 1+\alpha_0\Delta x$, where $\alpha_0 \ge 0$, then (UKC) is satisfied in $\Omega(\zeta_0')$ and $\dim \tilde{S}(0,1)\,\mathrm{Ker}\,\tilde{A} = 1$.

Theorem 5.2 is replaced by stronger theorem 5.3 and theorem 5.1 is formulated locally by means of estimate (6.8) with $|z_0| = 1+\alpha_0\Delta x$ and $\alpha_0 > 0$.

Let us consider the more complicated case $\mathrm{Re}\,\kappa_j'=0$ $(z_0' \neq 0)$. Using the variables $v(x) = X^{-1}(\zeta')u(x)$ and $G(x) = T^{-1}(\zeta')F(x)$ we arrive as in subsection 7.2 at equations (7.45). The columns of the matrix $X_{FI}(\zeta')$ as well as the components of the vectors $v_{FI}(x)$ and $G_{FI}(x)$ are partitioned in a natural way when $\mathrm{Re}\,\kappa_j' = 0$. Since the column $X_0$ is not included in the group I, equation (7.45) (C) should be rewritten as

$$(8.51)\quad \tilde{S}(\zeta)X_0(\zeta')v_0(0)+\tilde{S}(\zeta)X_I(\zeta')v_I(0)+\tilde{S}(\zeta)X_{II}(\zeta')v_{II}(0)+\tilde{S}(\zeta)X_\infty(\zeta')v_\infty(0) = g\,.$$

The symmetrizer $R(\zeta')$ is constructed as a block diagonal matrix, where the partial blocks are denoted according to the partition of the matrix $X(\zeta')$. We define $R_0(\zeta') = -crI$, where c is a small positive constant, and $R_\infty(\zeta') = R_\infty^{(1)}(\zeta')\oplus R_\infty^{(2)}(\zeta') = rI\oplus I$. If $\mathrm{Re}\,\kappa_j' = 0$, the blocks $R_j^{(1)}(\zeta')$ are defined as in subsection 8.2. If $\mathrm{Re}\,\kappa_j > 0$, then $R_j^{(1)}(\zeta') = I$ and for $\mathrm{Re}\,\kappa_j' < 0$, $R_j^{(1)}(\zeta') = -cI$. The matrices $R_k(\zeta')$, $k = 2,3,\ldots,n$, are defined similarly according to whether $|\kappa_k| > 1$ or $|\kappa_k| < 1$. Let us notethat for $\mathrm{Re}\,\kappa_j' \neq 0$ it follows that $\mathrm{Re}\,R_j^{(1)}(\zeta')M_j'(\zeta') \ge \delta I$ and therefore

$$(M_j^{(1)}(\zeta'))^* R_j^{(1)}(\zeta') M_j^{(1)}(\zeta') - R_j^{(1)}(\zeta') =$$

$$= (I + rM_j'(\zeta'))^* R_j^{(1)}(\zeta')(I + rM_j'(\zeta')) - R_j^{(1)}(\zeta') \geq \delta r\, I$$

for $r > 0$ sufficiently small. Since $r \geq |z-1| \geq |z| - 1$, estimate (8.32) holds for any $j = 1,2,\ldots,t$. So the symmetrizers $R_F(\zeta')$ and $R_\infty(\zeta')$ satisfy for any $\zeta' \in \Omega_R(\zeta_0')$ the conditions

$$(8.52)\quad M_F^*(\zeta')R_F(\zeta')M_F(\zeta') - R_F(\zeta') \geq \delta(|z|-1)I,\quad R_\infty(\zeta') - M_\infty^*(\zeta')R_\infty(\zeta')M_\infty(\zeta') \geq \delta(|z|-1)I\ ,$$

$$(8.53)\quad v_{F1}^* R_{F1}(\zeta')v_{F1} \geq -c|v_I|^2,\quad v_0^* R_0(\zeta')v_0 \geq -cr|v_0|^2,\quad v_\infty^* R_\infty(\zeta')v_\infty \geq r|v_\infty|^2.$$

Applying to equations (7.45) the generalized energy method as in subsection 7.2 we arrive at an estimate

$$(8.54)\quad \delta(|z|-1)\|v\|^2 + [|v_{II}(0)|^2 + |v_\infty^{(2)}(0)|^2 + r|v_\infty^{(1)}(0)|^2 - c(|v_I(0)|^2 + r|v_0(0)|^2]\Delta x$$

$$\leq K\frac{\|R(\zeta')G\|^2}{|z|-1}\ .$$

Since $r(T^{-1}(\zeta'))_0$, $r(T^{-1}(\zeta'))_\infty^{(1)}$ , $(T^{-1}(\zeta'))_\infty^{(2)}$ and $(T^{-1}(\zeta'))_{F1}$ are analytic in $\Omega(\zeta_0')$, it follows that

$$\| R(\zeta')G\|^2 \leq K\|F\|^2.$$

Analagously to lemma 7.7 we have

Lemma 8.10. The condition (UKC) in the neighbourhood $\Omega(\zeta_0')$ is equivalent to the condition $\det \tilde{S}(\zeta_0)(X_0(\zeta_0'),\ X_I(\zeta_0')) \neq 0$ .

Proof: Let us construct a block diagonal matrix $U_{F1}(\zeta')$ with partial blocks denoted as in the matrix $M_{F1}(\zeta')$. For $k = 2,3,\ldots,n$ and $j = 1,2,\ldots,t$ with

Re $\kappa_j' \neq 0$ we set $U_k$ and $U_j^{(1)}$ as unit matrices, and for Re $\kappa_j' = 0$ the matrix $U_j^{(1)}(\zeta')$ is defined as in (8.30). Then $U(\zeta') = \operatorname{diag}(U_0, U_{F1}(\zeta'), U_\infty) = \operatorname{diag}(U_F(\zeta'), U_\infty)$, where $U_0$ and $U_\infty$ are unite matrices of corresponding order. The matrix $U(\zeta')$ depends continuously on $\zeta'$ at the point $\zeta_0'$ with the value $U(\zeta_0')=I$ and $U_{F1}(\zeta')$ provides a similarity transformation

$$U_{F1}^{-1}(\zeta')M_{F1}(\zeta')U_{F1}(\zeta') = \begin{bmatrix} N_{11}(\zeta') & N_{12}(\zeta') \\ 0 & N_{22}(\zeta') \end{bmatrix} .$$

For $\zeta' \in \Omega_R(\zeta_0')$ the spectra of the matrices $N_{11}(\zeta')$ and $N_{22}(\zeta')$ lie correspondingly inside and outside the unit circle $|\kappa| = 1$. Considering the homogeneous equations (7.45) (A), (B) for $F = 0$ and performing a transformation $v = U(\zeta')y$, where the components of the vector $y$ are partitioned according to $v$, we arrive at the equations

$$E_x y_0 = 0$$

$$\left(E_x - \begin{bmatrix} N_{11} & N_{12} \\ 0 & N_{22} \end{bmatrix}\right) \begin{bmatrix} y_I \\ y_{II} \end{bmatrix} = 0 \tag{8.55}$$

$$(1 - M_\infty E_x) y_\infty = 0$$

Hence for $\zeta' \in \Omega_R(\zeta_0')$ the general homogeneous solution of equations (8.55) in $\ell_2(x)$ is given by

$$y_{II}(x_\nu) = y_\infty(x_\nu) = 0 \text{ for } \nu \geq 0,\ y_0(x_\nu) = 0 \text{ for } \nu \geq 1 \text{ and } y_I(x_\nu) = N_{11}^{\nu}(\zeta) y_I(0)$$

and the corresponding homogeneous solution of equation (7.44) (A) is

(8.56) $$\varphi(x_\nu,\zeta') = (\varphi_1(x_\nu,\zeta'),\dots,\varphi_n(x_\nu,\zeta'))(y_0(0),y_I(0))' =$$

$$X(\zeta')U(\zeta')(y_0(x_\nu),y_I(x_\nu),0)'.$$

The nm-dimensional vectors $\varphi_j(0,\zeta')$, $j = 1,2,\dots,n$, are continuous functions of $\zeta'$ at the point $\zeta_0'$. Since the matrix $X(\zeta')$ is non-singular (we, actually, use only the independence of columns $X_0(\zeta_0'),X_I(\zeta_0')$), it follows that the above vectors $\varphi_j(0,\zeta')$ are independent for any $\zeta'\in\Omega(\zeta_0')$ and, thus, may be used for the definition of the matrix $N(\xi,z)$ in (5.30). So the matrix $N(\xi,z) = N(\zeta')$ depends continuously on $\zeta'$ at the point $\zeta_0'$ and $N(\zeta_0') = \tilde{S}(\zeta_0)(X_0(\zeta_0'),X_I(\zeta_0'))$. The condition (UKC) obviously implies that $\det N(\zeta_0') \neq 0$. The converse is also true if one takes $\Omega(\zeta_0')$ small enough.

Let us return to the boundary condition (8.51). If (UKC) is fulfilled,we have an estimate

(8.57) $$|v_0(0)|^2 + |v_I(0)|^2 \leq K(|v_{II}(0)|^2 + |v_\infty(0)|^2 + |g|^2) .$$

Suppose that in addition $\dim \tilde{S}(\zeta_0)\operatorname{Ker}\tilde{A} = 1$. For $r = 0$ the columns of $(X_0(\zeta'),X_\infty^{(1)}(\zeta'))$ span the space $\operatorname{Ker}\tilde{A}$ and hence the columns of $\tilde{S}(\zeta_0)X_\infty^{(1)}(\zeta')$ depend linearly on $\tilde{S}(\zeta_0)X_0(\zeta') \neq 0$. Then for any $\zeta'\in\Omega(\zeta_0')$ there is an estimate

$$|v_I(0)|^2 \leq K(|v_{II}(0)|^2 + |v_\infty^{(2)}(0)|^2 + |rv_\infty^{(1)}(0)|^2 + |g|^2)$$

and therefore

(8.58) $$|v_I(0)|^2 + r|v_0(0)|^2 \leq K(|v_{II}(0)|^2+|v_\infty^{(2)}(0)|^2 + r|v_\infty^{(1)}(0)|^2 + |g|^2) .$$

Choosing the constant c in (8.54) to be small enough, one derives from (8.54) and (8.58):

$$\delta(|z|-1)\|v\|^2+(|v_{F1}(0)|^2+|v_\infty^{(2)}(0)|^2+r|v_0(0)|^2+r|v_\infty^{(1)}(0)|^2)\Lambda x\leq K(\frac{\|F\|^2}{|z|-1} + |g|^2\Lambda x) .$$

Since $\|u\|^2 = \|Xv\|^2 \leq K\|v\|^2$ and $\tilde{A}X_0(\zeta')$, $\tilde{A}X_\infty^{(1)}(\zeta') = O(r)$, we get the required estimate for theorem 5.3:

$$(|z|-1)\|u\|^2 + |\tilde{A}u(0)|^2\Delta x \leq K(\frac{\|F\|^2}{|z|-1} + |g|^2\Delta x) . \tag{8.59}$$

If only (UKC) is satisfied, it follows from (8.54) and (8.57) that

$$(|z|-1)\|v\|^2 + |v(0)|^2\Delta x \leq K(\frac{\|F\|^2}{|z|-1} + |g|^2\Delta x + |v_\infty^{(1)}(0)|^2\Delta x) . \tag{8.60}$$

The value of $v_\infty^{(1)}(0)$ is given by

$$v_\infty^{(1)}(0) = \sum_{\nu=0}^{m-1} (M_\infty^{(1)}(\zeta'))^\nu (T^{-1}(\zeta'))_\infty^{(1)} F(x_\nu)$$

since $M_\infty^{(1)}(\zeta')$ is a nilpotent Jordan cell of the order m-1. Therefore

$$|v_\infty^{(1)}(0)| \leq \frac{K|F_b|}{r} \leq \frac{K|F_b|}{|z|-1} \text{ , where } |F_b|^2 = \sum_{\nu=0}^{m-1} |F(x_\nu)|^2 .$$

If $|z_0| = 1 + \alpha_0\Delta x$ with $\alpha_0 > 0$, it follows for any $|z| > |z_0|$ that

$$\frac{\Delta x}{(|z|-1)^2} \leq \frac{K}{|z|-|z_0|} ,$$

and we arrive at the estimate

$$(|z|-|z_0|)\|u\|^2 + |u(0)|^2\Delta x \leq K(\frac{\|F\|^2+|F_b|^2}{|z|-|z_0|} + |g|^2\Delta x) \tag{8.61}$$

which is obviously stronger than (6.8).

It remains only to prove the necessity part of theorem 5.3. Suppose that (UKC) is not satisfied in $\Omega(\zeta_0')$, i.e. det $\tilde{S}(\zeta_0)(X_0(\zeta_0'), X_I(\zeta_0')) = 0$. There exists a non-zero vector $(y_0(0), y_I(0))'$ such that

$$\tilde{S}(\zeta_0)(X_0(\zeta_0')y_0(0) + X_I(\zeta_0')y_I(0)) = 0.$$

Using the vector $(y_0(0), y_I(0))$ we define by (8.56) a homogeneous solution $u(x) = \varphi(x,\zeta')$ of equation (7.44) (A). Then the vector $g = g(\zeta')$ in (7.44) depends continuously on $\zeta'$ when $\zeta' \to \zeta_0'$ and $g(\zeta_0') = 0$. Estimate (6.9) implies that

$$|\tilde{A}X(\zeta')U(\zeta')(y_0(0),y_I(0),0)'| \leq K|g(\zeta')|^2$$

and hence $\tilde{A}X_I(\zeta_0')y_I(0) = 0$. Since the columns of $X_I(\zeta_0')$ are independent of the space Ker $\tilde{A}$ , it follows that $y_I(0) = 0$. Therefore $y_0(0) \neq 0$ and $\tilde{S}(\zeta_0)X_0(\zeta_0') = 0$. Since $\tilde{S}$ and $X_0$ depend analytically on $\zeta$, it follows that $\tilde{S}(\zeta)X_0(\zeta')y_0(0) = g(\zeta') = O(r)$ and estimate (6.9) implies that

$$|y_0(0)|^2 \leq K|u(0)|^2 \leq \frac{K|u|^2}{\Delta x} \leq \frac{K|g|^2}{|z|-|z_0|} = \frac{O(r^2)}{|z|-|z_0|} .$$

Fixing $z' = z_0' + \varepsilon$ with small positive $\varepsilon$ and defining $z = 1 + rz'$ we obtain that $|z|-1 \geq r\varepsilon$. If $r$ and $\Delta x$ tend to zero in such a way that $|z|-|z_0| = |z|-1-\alpha_0\Delta x \geq r\varepsilon/2$, we obtain that $y_0(0) = 0$ and (UKC) follows.

In order to prove that dim $\tilde{S}(\zeta_0)$ Ker $\check{A} = 1$ let us assume first that Re $z_0'>0$. Consider equations (7.45) for $\zeta'\in\Omega(\zeta_0')$ with $g = 0$. Assume that the grid function $F(x)$ given in (7.44) vanishes for $x_\nu$ with $\nu \geq m$. Since the matrix $M_{F1}(\zeta')$ is partitioned into blocks $M_I(\zeta')$ and $M_{II}(\zeta')$, we may write for $r > 0$:

$$(8.62)\quad v_{II}(0) = -\sum_{\nu=0}^{m-1} M_{II}^{-\nu-1}(\zeta')(T^{-1}(\zeta'))_{II}F(x_\nu),\quad v_\infty(0) = \sum_{\nu=0}^{m-1} M_\infty^{\nu}(\zeta')(T^{-1}(\zeta'))_\infty F(x_\nu) .$$

The vectors $v_{II}(0)$ and $v_\infty(0)$ are functions of $\zeta'$ and the values of $v_I(0)$ and $v_0(0)$ may be found with the aid of the boundary condition (8.51). We denote $\hat{v}(0,\zeta') = rv(0)$. Since the matrix $\hat{T}^{-1}(\zeta') = rT^{-1}(\zeta')$ is analytic in $\Omega(\zeta_0')$ and (UKC) is satisfied, it follows that also $v(0,\zeta')$ is analytic. The analyticity of $(T^{-1}(\zeta'))_{F1}$ and $(T^{-1}(\zeta'))^{(2)}_{\infty}$ implies that $\hat v_{II}(0,\zeta')$, $\hat v^{(2)}_{\infty}(0,\zeta') = O(r)$. Since the last row of $(\hat T^{-1}(\zeta_0'))^{(1)}_{\infty}$ is non-zero and $M^{(1)}_{\infty}(\zeta')$ is a nilpotent Jordan cell, one can obtain any value of $\hat v^{(1)}_{\infty}(0,\zeta_0')$ by a suitable choice of $F(x)$. The vector $\hat v(0,\zeta_0')$ satisfies the boundary condition

$$(8.63)\quad \tilde S(\zeta_0)X_0(\zeta_0')\hat v_0(0,\zeta_0') + \tilde S(\zeta_0)X_I(\zeta_0')\hat v_I(0,\zeta_0') + \tilde S(\zeta_0)X^{(1)}_{\infty}(\zeta_0')\hat v^{(1)}_{\infty}(0,\zeta_0') = 0 .$$

Suppose that $\hat v_I(0,\zeta_0') \neq 0$. Since $\tilde A u(0) = \tilde A X(\zeta')\hat v(0,\zeta')/r$ and $\tilde A X_0(\zeta')$, $\tilde A X^{(1)}_{\infty}(\zeta')$ $= O(r)$, it follows that

$$|\tilde A u(0)| = |\tilde A X_I(\zeta')\hat v_I(0,\zeta')/r + O(1)| \geq \frac{\delta}{r} ,$$

where $\delta$ is a positive constant. Then the estimate

$$|\tilde A u_0|^2 \leq \frac{K\|F\|^2}{\Delta x(|z|-|z_0|)}$$

implies that

$$\frac{Kr^2}{|z|-|z_0|} \geq \delta > 0 \text{ for any } |z| - |z_0| = 1+\alpha_0\Delta x \text{ and any } \Delta x > 0,$$

which, as shown in the above proof of (UKC), is not true. This last contradiction means that $\hat v_I(0,\zeta_0') = 0$ and the vector $\tilde S(\zeta_0)X^{(1)}_{\infty}(\zeta_0')\hat v^{(1)}_{\infty}(0,\zeta_0')$ is proportional to $\tilde S(\zeta_0)X_0(\zeta_0')$. Since the columns of $(X_0(\zeta_0'),X^{(1)}_{\infty}(\zeta_0'))$ span the space Ker $\tilde A$ and $\hat v^{(1)}_{\infty}(0,\zeta_0')$ may accept any value, it follows that dim $\tilde S(\zeta_0)$Ker $\tilde A = 1$. If Re $z_0' = 0$, we can fix any $z_1'$ such that Re $z_1' > 0$ and $\zeta_1' = (\xi_0',z_1',0)\in\Omega(\zeta_0')$. Then there is some neighbourhood $\Omega(\zeta_1') \subset \Omega(\zeta_0')$, and estimate (6.9) holds in $\Omega(\zeta_1')$. So we prove the necessity part of theorem (5.3) for the neighbourhood $\Omega(\zeta_1')$.

Let us now turn to the case $z_0' = 0$. The operator I in estimate (6.9) should be defined as $P(\zeta') = \tilde P(\zeta)$, where $\zeta = \zeta'\cdot r$ . Theorems 5.1-5.4 are formulated locally in a neighbourhood $\Omega(\zeta_0')$ in a natural way. Let us define the symmetrizer $R(\zeta')$ as in the case $z_0' \neq 0$. Since there are no blocks $M_j'$ with $\operatorname{Re} \kappa_j' = 0$, we may write $r$ instead of $|z| - 1$ in (8.52). Therefore $|z| - 1$ in (8.54) should be replaced by $r$, so that we obtain

$$(8.64)\quad \delta r\|v\|^2+[|v_{II}(0)|^2+|v_\infty^{(2)}(0)|^2+r|v_\infty^{(1)}(0)|^2-c(|v_I(0)|^2+r|v_0(0)|^2]\Delta x \leq K\frac{\|R(\zeta')G\|^2}{r}$$

Since $rz'(T^{-1}(\zeta'))_0$, $rz'(T^{-1}(\zeta'))_\infty^{(1)}$, $z'(T^{-1}(\zeta'))_{FI}^{(1)}$ and $(T^{-1}(\zeta'))^{(2)}$ are analytic in $\Omega(\zeta_0')$, it follows that $\|R(\zeta')G\|^2 \leq K\|F\|^2/|z'|^2$. Lemma 8.10 is now proved easily since the matrix $M_{FI}(\zeta')$ is partitioned into the blocks $M_I(\zeta')$ and $M_{II}(\zeta')$. We should only recall that the columns of the matrix $(X_0(\zeta_0'), X_I(\zeta_0'))$ are independent according to part (c) of lemma 8.6. If (UKC) is satisfied, we have estimate (8.57), and if additionally $\dim \tilde{\check{S}}(\zeta_0)\operatorname{Ker} \tilde{A} = 1$, also (8.58) holds. So in the last case instead of (8.59) we obtain an estimate

$$(8.65)\qquad \|u\|^2 + \frac{|\check{A}u(0)|^2\Delta x}{r} \leq K\left(\frac{|g|^2\Delta x}{r} + \frac{\|F\|^2}{|z-1|^2}\right)$$

which is obviously stronger than estimate (6.7) for $|z_0| = 1$. If only (UKC) is satisfied, we get instead of (8.60) an estimate

$$r\|v\|^2 + |v(0)|^2\Delta x \leq K\left(\frac{\|F\|^2}{r|z'|^2} + |g|^2\Delta x + |v_\infty^{(1)}(0)|^2\Delta x\right),$$

where

$$|v_\infty^{(1)}(0)| \leq \frac{K|F_b|}{|rz'|} \leq \frac{K|F_b|}{|z|-1}.$$

Then estimate (6.8) with $|z_0| = 1+\alpha_0\Delta x > 1$ follows as in the case $z_0' \neq 0$.

In order to prove the sufficiency part of theorem 5.3 let us introduce grid vector functions $\acute{v}(x)$ and $\acute{G}(x)$ whose components are partitioned according to $v(x)$ and $G(x)$ and are given by:

$$\acute{v}_0 = rz'v_0,\ \acute{v}_\infty^{(1)} = rz'v_\infty^{(1)},\ \acute{v}_{FI}^{(1)} = z'v_{FI}^{(1)},\ \acute{v}^{(2)} = v^{(2)}$$

and $\hat{G}(x)$ is expressed in terms of $G(x)$ in the same way. Equations (7.49) (A), (B) remain unchanged in the new variables:

$$(8.66)\qquad \begin{aligned} &(A)\quad (E_x - M_F(\zeta'))\hat{v}_F(x) = \hat{G}_F(x) \\ &(B)\quad (I - M_\infty(\zeta')E_x)\hat{v}_\infty(x) = \hat{G}_\infty(x) \end{aligned}$$

Let us modify the former symmetrizer $R(\zeta')$ by changing $R_0$ and $R_\infty^{(1)}$ from $-crI$ and $rI$ to $-cI$ and $I$ respectively. Applying to the above equations the generalized energy method with the modified symmetrizer we get instead of (8.64) an estimate

$$(8.67)\qquad \delta r\|\hat{v}\|^2 + [|\hat{v}_{II}(0)|^2 + |\hat{v}_\infty(0)|^2 - c(|\hat{v}_I(0)|^2 + |\hat{v}_0(0)|^2)]\Delta x \le \frac{K\|\hat{G}\|^2}{r} .$$

Since det $\tilde{S}(\zeta_0)(X_0(\zeta_0'), X_I(\zeta_0')) \neq 0$, the vectors $v_0(0)$ and $v_I(0)$ in (8.61) are linear functions of $v_{II}(0)$, $v_\infty(0)$ and $r$ with coefficients [illegible] $\zeta_0'$. If $r = 0$, the vector $\tilde{S}(\zeta_0)X_\infty^{(1)}(\zeta')v_\infty^{(1)}(0) \in$ [illegible] Ker $\hat{A}$ [illegible] to $\tilde{S}(\zeta_0)X_0(\zeta')$, and therefore

$$v_I(0) = O(rv_\infty^{(1)} [illegible], v_{II} [illegible], v_\infty [illegible], r [illegible]).$$

If $z' = 0$, then by part b) of lemma [illegible] the columns of [illegible] $(\zeta')$, $X$ [illegible] $(\zeta')$, $X_\infty^{(1)}(\zeta')$) form a basis of Ker $\tilde{S}(\zeta)$ and, [illegible], the [illegible] columns of $\tilde{S}(\zeta)(X_0(\zeta'), X_I(\zeta')$ [illegible]. Hence

$$v_I^{(1)}(0) = [illegible]$$

and using the previous [illegible] of $v_\infty^{(1)}(0)$ in the expression [illegible]

$$|v_{II} [illegible]| + |v_\infty [illegible]$$

[illegible] the constant [illegible]

$$|\hat{v}(0)|^2\Delta x \le K\left(\frac{\|\hat{G}\|^2}{r} + |g|^2\Delta x\right).$$

Let us note that according to part a) of lemma 8.7 it follows that $\|\hat{G}\| \le K\|F\|$. On the other hand

$$\tilde{P}(\xi)u(0)=\tilde{P}(\xi)(X_0(\zeta')v_0(0)+X_\infty^{(1)}(\zeta')v_\infty^{(1)}(0)+X_{F1}^{(1)}(\zeta')v_{F1}^{(1)}(0)+X^{(2)}(\zeta')v^{(2)}(0)).$$

Since $\tilde{P}(\xi)X_0(\zeta') = \tilde{P}(\xi)X_\infty^{(1)}(\zeta') = 0$ if $rz' = 0$ and $\tilde{P}(\xi)X_{F1}^{(1)}(\zeta') = 0$ if $z' = 0$, it follows that $|\tilde{P}(\xi)u(0)| \le K|\hat{v}(0)|$ and therefore

$$\frac{|\tilde{P}(\xi)u(0)|^2\Delta x}{|z|-1} \le K\left(\frac{\|F\|^2}{(|z|-1)^2} + \frac{|g|^2\Delta x}{|z|-1}\right).$$

The last result together with (8.65) gives us the estimate (6.9).

Suppose now that estimate (6.9) is satisfied in $\Omega(\zeta_0')$ with $P(\zeta)=\tilde{P}(\xi)$ and $|z_0| = 1+\alpha\Delta x \ge 0$. We shall show that (UKC) is then fulfilled. Otherwise there exists a non-zero vector $(v_0(0),v_I(0))'$ such that

$$\tilde{S}(\zeta)(X_0(\zeta')v_0(0) + X_I(\zeta')v_I(0)) = g(\zeta') \text{ and } g(\zeta_0') = 0.$$

Let $v(x)$ be the solution of equations (7.45) (A), (B) for $F = 0$ and $\zeta'\in\Omega_R(\zeta_0')$ corresponding to the above $v_0(0)$, $v_I(0)$. Since the matrix $M_{F1}(\zeta')$ is partitioned into blocks $M_I(\zeta')$ and $M_{II}(\zeta')$, it follows that $v_{II}(x) = v_\infty(x) = 0$. The columns of the matrix $(X_0(\zeta'),X_I(\zeta'))$ are independent for any $\zeta'\in\Omega(\zeta_0')$. Therefore $\|u\| = \|X_0(\zeta')v_0 + X_I(\zeta')v_I\| \approx \|v\|$. Estimate (6.9) implies that

$$\|v\|^2 + \Delta x|\tilde{P}(\xi)\ X(\zeta')v(0)|^2/(|z|-|z_0|) \le K\Delta x|g(\zeta')|^2/(|z|-|z_0|). \tag{8.68}$$

Since $\tilde{P}(0)X^{(1)}(\zeta_0') = 0$ and the columns of $\tilde{P}(0)X^{(2)}(\zeta')$ are independent, it follows that $v_I^{(2)}(0) = 0$. Let us estimate the term $\|v_I^{(1)}\|^2$. Since $M_I^{(1)}(\zeta') = I + O(r)$, we have for any vector w an estimate $|M_I^{(1)}(\zeta')\varphi| \geq (1-Kr)|\varphi|$. Hence $\|v_I^{(1)}\|^2 \geq \delta\Delta x|v_I^{(1)}(0)|^2/r$, and (8.68) implies that

$$|v_I^{(1)}(0)|^2 \leq Kr|g(\zeta')|^2/(|z|-|z_0|).$$

Let us set $\zeta' = (\xi_0', r, z'=r)$ with $r > 0$. Then $|g(\zeta')|^2 = |O(\zeta'-\zeta_0')|^2 = O(r^2)$ and $|z|-|z_0| = r^2-\alpha_0\Delta x$. When r and $\Delta x$ tend to zero in such a way that $|\alpha_0\Delta x| \leq r^2/2$, we obtain from the last estimate that $v_I^{(1)}(0) = 0$. It remains only to show that $\tilde{\check{S}}(\zeta_0)X_0(\zeta_0') \neq 0$. But the vector $X_0(\zeta')$ depends actually on $\zeta$ so that for $\zeta_1' = (\xi_0', 0, z')$, $X_0(\zeta_1') = X_0(\zeta_0')$. Taking some point $\zeta_1' \in \Omega(\zeta_0')$ with Re $z' > 0$ and its neighbourhood $\Omega(\zeta_1') \subset \Omega(\zeta_0')$, we prove as in the case $z_0' \neq 0$ that $\tilde{\check{S}}(\zeta_0)X_0(\zeta_1') \neq 0$.

Let us now fix a point $\zeta_1' = (\xi_0', r, z'=0) \in \Omega(\zeta_0')$ with $r > 0$. Then there is a small neighbourhood $\Omega(\zeta_1)$ of the point $\zeta_1 = (\xi_0' \cdot r, z=1)$ such that for any $\zeta \in \Omega(\zeta_1)$ the corresponding point $\zeta'$ belongs to $\Omega(\zeta_0')$. Since (6.9) holds in $\Omega(\zeta_0')$, it holds also for $\zeta = (\xi, z) \in \Omega(\zeta_1)$ with $|z| > 1$. According to the local version of theorem 5.3 proved in subsection 7.2, we conclude that dim $\tilde{\check{S}}(\zeta_1)$Ker $\check{P}(\xi_0' \cdot r) = (n+1)/2$. It follows then from the considerations of continuity that dim $\tilde{\check{S}}(\zeta_0)$Ker $\tilde{\check{P}}(0) \leq (n+1)/2$. But the $(n+1)/2$ columns of $\tilde{\check{S}}(\zeta_0)(X_0(\zeta_0'), X_I^{(1)}(\zeta_0'))$ belong to $\tilde{\check{S}}(\zeta_0)$Ker $\check{P}(0)$ and are independent according to (UKC). Therefore dim $\tilde{\check{S}}(\zeta_0)$Ker $\tilde{\check{P}}(0) = (n+1)/2$.

It remains now to show that dim $\tilde{\check{S}}(\zeta_0)$ Ker $\hat{A} = 1$. Since we prove theorem 5.3 locally, we can not refer to the case $z_0' \neq 0$ where F was set equal to $\hat{A}$. However, the proof is similar. Let us take in (7.44) a grid function F(x) vanishing for $x_\nu > m\Delta x$ and set $g = 0$. Then the corresponding values of

$v_{II}(0), v_\infty^{(1)}(0)$ and $v_\infty^{(2)}(0)$ in (8.62) are of order $O(1/z')$, $O(1/(rz'))$ and $O(1)$ respectively. Therefore the vector $\hat{v}(0,\zeta') = rz'v(0)$ depends analytically on $\zeta' \in \Omega(\zeta_0')$ with $\hat{v}_{II}(0,\zeta') = O(r)$ and $\hat{v}_\infty^{(2)}(0,\zeta') = O(rz')$. Suppose that condition 5.2 is not satisfied. Choosing suitable $F(x)$ one can assume that the vector $\tilde{S}(\zeta_0)X_\infty^{(1)}(\zeta_0')\hat{v}_\infty^{(1)}(\zeta_0')$ in (8.63) is not proportional to $\tilde{S}(\zeta_0)X_0(\zeta_0')$, and hence $v_I(0,\zeta_0') \neq 0$. We have already proved that the columns of $\tilde{S}(\zeta_0)(X_0(\zeta_0'), X_I^{(1)}(\zeta_0'))$ form a basis of the space $\tilde{S}(\zeta_0)$ (Ker $\tilde{P}(0)$). Since the vector $X_\infty^{(1)}(\zeta_0')\hat{v}_\infty^{(1)}(\zeta_0')$ belongs to Ker $\tilde{A} \subset$ Ker $\tilde{P}(0)$, it follows that $\hat{v}_I^{(1)}(0,\zeta_0') \neq 0$. We have an estimate

$$\|v_I^{(1)}\|^2 \geq \frac{\delta\Delta x|v_I^{(1)}(0)|^2}{r} - \frac{K\|(T^{-1}(\zeta'))_I^{(1)}F\|^2}{r^2}$$

(8.69)

$$\geq \frac{\delta\Delta x|\hat{v}_I^{(1)}(0,\zeta')|^2}{r|z-1|^2} - \frac{K\|F\|^2}{|z-1|^2} \geq \frac{\delta_1\Delta x}{r|z-1|^2} .$$

Since the norms of the remaining components of v are of smaller order, it follows that $\|u\|^2 \geq \dfrac{\delta\Delta x}{r|z-1|^2}$. But the last estimate contradicts the estimate $\|u\|^2 \leq \dfrac{K\|F\|^2}{(|z|-1)^2}$ for positive z. Thus, theorem 5.3 is completely proved.

[illegible] one can attain the same results [illegible] finite covering by neighbourhoods of the type [illegible].

## [illegible] structure of the κ-matrix $\overset{\sim}{L}(\kappa,\zeta)$ in a [illegible] $\zeta_1' \neq$ [illegible]

According to statement 6.4 the characteristic equation [illegible] root $\kappa = -1$ of multiplicity $2n-2$ and a simple root $\kappa_0 =$ [illegible]. [illegible] (6.19) has for any $\zeta \in \Omega(\zeta_0)$ a root $\kappa =$ [illegible] multiplicity $(m-2)n+1$. [illegible] root $\kappa_\infty = \infty$ is an eigenvalue of $\overset{\sim}{L}(\kappa,\zeta)$ of the [illegible] multiplicity. In order to describe the roots $\kappa$ near $\kappa = -1$ as $\zeta$ tends to $\zeta_0$ we introduce $\kappa'$-mat- [illegible] as in subsection 8.1

$$[illegible] = [illegible](\kappa',\zeta' [illegible] \kappa,\zeta [illegible] + [illegible], \text{where}$$

$$\alpha' = \alpha'(\kappa',\zeta') = (\kappa-1)\sin(\xi' r/2)/r, \quad \beta' = \beta'(\kappa',\zeta') = i\kappa'\sin(\xi/2), \tag{9.3}$$

and

$$L'(\kappa',\zeta') = L(\kappa,\zeta)/r^2 = z'\kappa + \frac{C'(\kappa',\zeta')}{2}(\kappa'\sin(\xi' r/2) - C'(\kappa',\zeta'))$$

$$= \ell(C',\kappa',\zeta'),$$

where $\ell(C',\kappa',\zeta')$ is considered as a polynomial of second degree in $C'$ with coefficients depending on $\kappa',\zeta'$. The values of $\zeta,\kappa$ in (9.3) are given by (9.2). For $r = 0$ we obtain

$$C'(\kappa',\zeta') = -A\xi' + Bi\kappa', \text{ and } L'(\kappa',\zeta') = -z' - (C'(\kappa',\zeta'))^2/2 . \tag{9.4}$$

Since $|L'(\kappa',\zeta')| \equiv 0$ when $z'\kappa = 0$, it follows that $|L'(\kappa',\zeta')| = z'\kappa p(\kappa',\zeta')$, where $p(\kappa',\zeta')$ is a polynomial in $\kappa'$ depending analytically on $\zeta'$. Hence the characteristic equation for the $\kappa'$-matrix $L'(\kappa',\zeta')$, when $\kappa'$ is bounded and $r$ is small (so that $\kappa = -1 + r\kappa' \neq 0$), *is equivalent to the equation*

$$p(\kappa',\zeta') = 0 . \tag{9.5}$$

The matrix $L'(\kappa',\zeta')$ may be decomposed as

$$L'(\kappa',\zeta') = -(1/2)(s_1' I + C')(s_2' I + C') , \tag{9.6}$$

where $-s'_{1,2}$ are roots of the quadratic equation $\ell(s',\kappa',\zeta') = 0$. If $z_0' \neq 0$, $s_1'$ and $s_2'$ *depend analytically* on $\kappa'$ and $\zeta' \in \Omega(\zeta_0')$, and

$$s'_{1,2}(\kappa',\zeta_0') = s'_{1,2}(\zeta_0') = \pm\sqrt{-2z_0'} .$$

It is clear that Im $s'_{1,2}(\zeta_0') \neq 0$ for $z_0' \neq 0$ and Re $z_0' \geq 0$. We shall suppose in that case that Im $s_1'(\zeta_0') > 0$ and therefore Im $s_2'(\zeta_0') < 0$. For $z_0' = 0$, $s_1'$ and $s_2'$ are continuous functions of $\kappa'$ and $\zeta'$ at the point $(\kappa',\zeta_0')$

and $s'_{1,2}(\zeta'_0) = 0$. According to formula (3.3)

$$|s'I+C'| = s'p_0(\alpha',\beta',s') .$$

Then using (9.6) and the fact that $s'_1 \cdot s'_2 = -2z'\kappa$, we obtain for $z' \neq 0$

$$p(\kappa',\zeta') = \text{const. } p_0(\alpha',\beta',s'_1)p_0(\alpha',\beta',s'_2) . \tag{9.7}$$

It follows from the continuity considerations that (9.7) is also valid for $z' = 0$. Thus, for $\zeta' = \zeta'_0$ even in the case $z'_0 = 0$ equation (9.5) may be written as

$$p_0(-\xi'_0,i\kappa',s'_1(\zeta'_0))p_0(-\xi'_0,i\kappa',s'_2(\zeta'_0)) = 0 . \tag{9.8}$$

Since the $\kappa'$-polynomial $p_0(-\xi'_0,i\kappa',s')$ is regular for any values of $\xi'_0$ and $s'$, the polynomial $p(\kappa',\zeta'_0)$ is also regular in $\kappa'$. As in statement 3.1 one can show that for $s' = 0$ or $\mathrm{Im}\, s' \neq 0$ the equation $p_0(-\xi'_0,i\kappa',s') = 0$ has $(n-1)/2$ roots $\kappa'$ with $\mathrm{Re}\, \kappa' > 0$ and the same number of roots with $\mathrm{Re}\, \kappa' < 0$. Therefore equation (9.8) has no imaginary roots $\kappa'$, and the difficulties associated with constructing the symmetrizer in subsection 8.2 do not appear here. Let $\kappa'_1,\kappa'_2,\ldots,\kappa'_t$ be the different roots of equation (9.8) with multiplicities $q_1^{(1)},q_2^{(1)},\ldots,q_t^{(1)}$. It is clear that $\sum_{j=1}^{t} q_j^{(1)} = 2n-2$. As in subsection 8.1 we select small neighbourhoods $\Omega(\kappa'_j)$ of the points $\kappa'_j$, $j = 1,2,\ldots,t$, and circular contours $\Gamma'_j \subset \Omega(\kappa'_j)$ bounding other neighboruhoods $\Omega_0(\kappa'_j)$. Then $\Omega(\zeta'_0)$ is set small enough so that any root $\kappa'$ of equation (9.5) belongs for $\zeta'\in\Omega(\zeta'_0)$ to some $\Omega_0(\kappa'_j)$. For any $\zeta'\in\Omega(\zeta'_0)$ with $z'r \neq 0$ we define as in (8.8) mutually orthogonal projectors

$$P_j(\zeta') = (2\pi i)^{-1} \oint_{\kappa'\in\Gamma_j'} \tilde{L}^{-1}(\kappa,\zeta)\tilde{A}_1(\zeta)d\kappa \quad , \qquad j = 1,2,\dots,t$$

$$P_0(\zeta) = (2\pi i)^{-1} \int_{\kappa\in\Gamma_0} \tilde{L}^{-1}(\kappa,\zeta)\tilde{A}_1(\zeta)d\kappa \tag{9.9}$$

$$P_\infty(\zeta) = (2\pi i)^{-1} \oint_{\kappa\in\Gamma_0} (\tilde{L}^{(\infty)}(\kappa,\zeta))^{-1}\tilde{A}_0(\zeta)d\kappa$$

Here as before $\Gamma_0$ is a contour around $\kappa_0 = 0$. For $j = 1,2,\dots,t$, we can write

$$P_j(\zeta') = (2\pi i)^{-1}r^{-1} \oint_{\kappa'\in\Gamma_j'} F(\kappa)[L'(\kappa',\zeta')^{-1}\oplus 0_{(m-1)n}]E(\kappa,\zeta)\tilde{A}_1(\zeta)d\kappa' . \tag{9.10}$$

Now unlike $P_j^{(1)}(\zeta')$ in (8.9) each of the projectors $P_j(\zeta')$ has a singularity of the type $r^{-1}$ even in the neighbourhood $\Omega(\zeta_0')$ with $z_0' \neq 0$. However, the projectors $P_0(\zeta)$ and $P_\infty(\zeta)$ have similar features as in Section 8.

Lemma 9.1 a) *There exist matrix valued functions* $X_0(\zeta)$ and $X_\infty(\zeta) = (X_\infty^{(1)}(\zeta), X_\infty^{(2)}(\zeta))$ analytic in $\Omega(\zeta_0)$ whose columns are independent for any $\zeta\in\Omega(\zeta_0)$ and form for $z \neq 1$ a basis of the spaces Im $P_0(\zeta)$ and Im $P_\infty(\zeta)$ respectively.

b) $X_0(\zeta)$ is one column matrix and consists of the singular eigenvector $\tilde{\varphi}_0(0,\xi)$. The columns of $X_\infty^{(1)}(\xi,z)$ form a singular Jordan chain of length m-1 corresponding to the eigenvalue $\kappa = 0$ of $\tilde{L}^{(\infty)}(\kappa,\xi,1)$; this chain is generated by the singular root function $\tilde{\varphi}_0^{(\infty)}(\kappa,\xi)$ at the point $\kappa = 0$.

c) The columns of the matrix $(X_0(\pi,z),X_\infty^{(1)}(\pi,z))$ form a basis of the space Ker $\tilde{B}$, where $\tilde{B} = \mathrm{diag}(B,B,\dots,B)$. The columns of $X_\infty^{(2)}(\zeta_0)$ form a basis of the space Im diag$(0,0,B,B,\dots,B)$ and are independent of the space Ker $\tilde{P}(\pi)$.

d) There are matrix-valued functions $M_0(\zeta) \equiv 0$ and $M_\infty(\zeta) = M_\infty^{(1)}(\zeta)\oplus M_\infty^{(2)}(\zeta)$ analytic in $\Omega(\zeta_0)$ satisfying the identities

$$\begin{aligned} &\tilde{A}_1(\zeta)X_0(\zeta)M_0(\zeta) + \tilde{A}_0(\zeta)X_0(\zeta) = 0 \\ &\tilde{A}_0(\zeta)X_\infty(\zeta)M_\infty(\zeta) + \tilde{A}_1(\zeta)X_\infty(\zeta) = 0 \end{aligned} \tag{9.11}$$

and $M_\infty(\zeta)$ is a Jordan matrix with eigenvalue $\kappa = 0$.

The proof is a word for word repetition of the one used in lemma 8.2 and is, therefore, omitted.

Consider now the projectors $P_j(\zeta')$ in (9.10). As in (8.16) we define an operator $Q_j(\zeta')$: $\Phi(\Omega(\kappa_j')) \to \mathbb{C}^{mn}$ by

$$Q_j(\zeta')\varphi = (2\pi i)^{-1} \oint_{\Gamma_j'} F_1(\kappa)L'(\kappa',\zeta')^{-1}\varphi(\kappa')d\kappa' . \tag{9.12}$$

For $rz' \neq 0$ the images of $Q_j(\zeta')$ and $P_j(\zeta')$ coincide. If $z_0' \neq 0$, we can write

$$\begin{aligned} Q_j(\zeta_0')\varphi = (2\pi i)^{-1}F_1(-1) \oint_{\Gamma_j'} &(s_1'(\zeta_0')I + C'(\kappa',\zeta_0'))^{-1} \times \\ &\times (s_2'(\zeta_0')I + C'(\kappa',\zeta_0'))^{-1}\varphi(\kappa')d\kappa' . \end{aligned} \tag{9.13}$$

The roots $\kappa_j'$ of equation (9.8) may be divided into two groups I and II according to whether Re $\kappa_j' > 0$ or Re $\kappa_j' < 0$. Each group consists of n-1 elements. Let $\Gamma_I$ be a contour in the halfplane Re $\kappa' > 0$ surrounding all the points of the group I and analogously the contour $\Gamma_{II}$ in the half plane Re $\kappa' < 0$ surrounds the points of the group II. Define the projectors

$$\begin{aligned} P_I^{(1)}(\zeta_0') &= (2\pi i)^{-1} \oint_{\Gamma_I} (s_1'(\zeta_0')I + C'(\kappa',\zeta_0'))^{-1}Bid\kappa' \\ P_I^{(2)}(\zeta_0') &= (2\pi i)^{-1} \oint_{\Gamma_I} (s_2'(\zeta_0')I + C'(\kappa',\zeta_0'))^{-1}Bid\kappa' \end{aligned} \tag{9.14}$$

and similarly $P_{II}^{(1)}(\zeta_0'($ and $P_{II}^{(2)}(\zeta_0')$.

Suppose that $\xi_0' = 0$, and hence $C'(\kappa',\zeta_0') = Bi\kappa'$. Then the image of $P_I^{(1)}(\zeta_0')$ is spanned by the eigenvectors of the matrix B corresponding to its negative eigenvalues and Im $P_I^{(2)}(\zeta_0')$ is spanned by those eigenvectors which corresponding to to the positive eigenvalues. Therefore we obtain a decomposition of the space $\mathbb{C}^n$ in a direct sum

$$\text{Im } P^{(1)}(\zeta_0') \oplus \text{Im } P^{(2)}(\zeta_0') \oplus \text{Ker } B = \mathbb{C}^n \tag{9.15}$$

and similarly

$$\text{Im } P_{II}^{(1)}(\zeta_0') \oplus \text{Im } P_{II}^{(2)}(\zeta_0') \oplus \text{Ker } B = \mathbb{C}^n \; . \tag{9.16}$$

One can consider the projectors $P_{I,II}^{(1),(2)}$ in (9.14) as homogeneous functions of zero order depending on free variables $\xi_0'$ and $s_1'$, where $s_2'$ in the expression for $P_{I,II}^{(2)}$ is replaced by $-s_1'$. Let D be any compact linearly connected set in $\mathbb{C}^2$ consisting of points $(\xi_0',s_1')$ with real $\xi_0'$ and Im $s_1' > 0$ and including a point $(0,s_1')$. One can select the contours $\Gamma_I$ and $\Gamma_{II}$ in such a way that no root $\kappa'$ of the equation $p_0(-\xi_0',i\kappa',s_1')p_0(-\xi_0',i\kappa',-s_1') = 0$ will cross these contours when $(\xi_0',s_1') \in D$. Then the projectors $P_{I,II}^{(1),(2)}$ depend *analytically* on $(\xi_0',s_1') \in D$ and, thus, equalities (9.15), (9.16) hold for all but a finite number of the fractions $\xi_0'/s_1'$. Since for $z_0' \neq 0$, Re $z_0' \geq 0$ and real $\xi_0'$ the point $(\xi_0',s_1'(\zeta_0'))$ may be included in such domain D, it follows that (9.15) and (9.16) hold for all, except possibly a finite number of the fractions $\xi_0'/s_1'(\zeta_0')$. Let us now formulate

Assumption 9.1. Equalities (9.15), (9.16) hold for any point $\zeta_0'$ with real $\xi_0'$ and Re $z_0' \geq 0$, $z_0' \neq 0$.

It *may* be easily verified that this assumption is satisfied in the case of the acoustic equations. We shall, actually, *not* use this assumption in the study of the block structure of the matrix $\tilde{L}(\kappa,\zeta)$ and only apply it in subsection 9.3 for the proof of theorems 5.1-5.3.

Let $\kappa_j'$ be a root of the polynomials $p_0(-\xi_0', i\kappa', s_1'(\zeta_0'))$ and $p_0(-\xi_0', i\kappa', s_2'(\zeta_0'))$ of multiplicities $q_j^{(1)}$ and $q_j^{(2)}$ respectively (only one of this multiplicities may be zero). Define operators

$$(9.17)\quad Q_j^{(1)}(\zeta')\varphi = (2\pi i)^{-1} \oint_{\Gamma_j'} F_1(\kappa)(s_1'(\kappa',\zeta')I + C'(\kappa',\zeta'))^{-1}\varphi(\kappa')d\kappa' ,$$

$$(9.18)\quad Q_j^{(2)}(\zeta')\varphi = (2\pi i)^{-1} \oint_{\Gamma_j'} F_1(\kappa)(s_2'(\kappa',\zeta')I + C'(\kappa',\zeta'))^{-1}\varphi(\kappa')d\kappa' .$$

Let us rewrite (9.17) in a form

$$(9.19)\quad Q_j^{(1)}(\zeta')\varphi = (2\pi i)^{-1} \oint_{\Gamma_j'} F(\kappa)[(s_1'(\kappa',\zeta')I + C'(\kappa',\zeta'))^{-1} \oplus I_{(m-1)n}]\varphi(\kappa')d\kappa' .$$

Supposing that $\kappa = -1 + \kappa' r$, we denote by $L_1^{-1}(\kappa',\zeta')$ the whole nmxnm matrix in the last integral. For $r \neq 0$ the matrix $\tilde{L}(\kappa,\zeta)$ with $\kappa$ and $\zeta$ given by (9.2) may be considered as a linear regular $\kappa'$-matrix. Since the $\kappa'$-matrix $L_1(\kappa',\zeta')$ is a right divisor of $\tilde{L}(\kappa,\zeta)$ and has $q_j^{(1)}$ eigenvalues $\kappa'$ in $\Omega_0(\kappa_j')$, it follows by remark 2.5 that

$$(9.20)\qquad \dim \operatorname{Im} Q_j^{(1)}(\zeta') = q_j^{(1)} .$$

The image of $Q_j^{(1)}(\zeta_0')$ is isomorphic to the image of the operator

$$\varphi(\kappa') \to (2\pi i)^{-1} \oint_{\Gamma_j'} (s_1'(\zeta_0')I + C'(\kappa',\zeta_0'))^{-1}\varphi(\kappa')d\kappa'$$

and has, therefore, the dimension $q_j^{(1)}$. Hence, (9.20) holds for any $\zeta' \in \Omega(\zeta_0')$. Similarly we have

(9.21) $$\dim \operatorname{Im} Q_j^{(2)}(\zeta') = q_j^{(2)}.$$

It is clear that $\operatorname{Im} Q_j^{(1)}(\zeta')$ and $\operatorname{Im} Q_j^{(2)}(\zeta')$ belong to $\operatorname{Im} Q_j(\zeta')$. Substituting the representation

$$\varphi(\kappa')=(s_1'(\kappa',\zeta')-s_2'(\kappa',\zeta'))^{-1}[(s_1'(\kappa',\zeta')I+C'(\kappa',\zeta'))-(s_2'(\kappa',\zeta')I+C'(\kappa',\zeta'))]\varphi(\kappa')$$

in (9.12) we conclude that

(9.22) $$\operatorname{Im} Q_j(\zeta') = \operatorname{Im} Q^{(1)}(\zeta') + \operatorname{Im} Q^{(2)}(\zeta') \ .$$

For $r \neq 0$ the space $\operatorname{Im} Q_j(\zeta')$ is of the dimension $q_j = q_j^{(1)} + q_j^{(2)}$ and, hence, the above sum is direct. For $\operatorname{Re} \kappa_j' > 0$ we have inclusions $\operatorname{Im} Q_j^{(1)}(\zeta_0') \subset F_1(-1) \operatorname{Im} P_I^{(1)}(\zeta_0')$, $\operatorname{Im} Q_j^{(2)}(\zeta_0') \subset F_1(-1) P_I^{(2)}(\zeta_0')$. Therefore if (9.15) and (9.16) hold at the point $\zeta_0'$, the sum in (9.22) is direct for any point $\zeta'$ of sufficiently small neighbourhood $\Omega(\zeta_0')$. It follows from (9.20) that there exists an $n x q_j^{(1)}$ matrix $\Psi(\kappa')$ analytic in $\Omega(\kappa_j')$ such that the columns of the matrix $X_j^{(1)}(\zeta') = Q_j^{(1)}(\zeta')(\Psi(\kappa'))$ form a basis of the space $\operatorname{Im} Q_j^{(1)}(\zeta')$ for any $\zeta' \in \Omega(\zeta_0')$. Since the whole integrand in (9.17) being multiplied on the left by $\tilde{L}(\kappa,\zeta)$ is analytic in $\Omega(\kappa_j')$ as a function of $\kappa'$, we obtain an identity

$$\tilde{A}_1(\zeta) Q_j^{(1)}(\zeta')(\kappa\Psi(\kappa')) + \tilde{A}_0(\zeta) X_j^{(1)}(\zeta') = 0 \ .$$

Then expressing $Q_j^{(1)}(\zeta')(\kappa'\Psi(\kappa')) = X_j^{(1)}(\zeta') M_j^{(1)}(\zeta')$, where $M_j^{(1)}(\zeta')$ is analytic in $\Omega(\zeta_0')$, we arrive at

$$\tilde{A}_1(\zeta) X_j^{(1)}(\zeta')(-I + r M_j^{(1)}(\zeta')) + \tilde{A}_0(\zeta) X_j^{(1)}(\zeta') = 0 \ .$$

As in subsection 8.1 it may be shown that the matrix $M_j^{(1)}(\zeta_0')$ has the only eigenvalue $\kappa_j'$ of multiplicity $q_j^{(1)}$. Similarly, one can define the matrices $X_j^{(2)}(\zeta')$ and $M_j^{(2)}(\zeta')$ for the operator $Q_j^{(2)}(\zeta')$. Denote

$$M_j' = M_j^{(1)} \oplus M_j^{(2)} \ , \ M_j = -I + rM_j' \ , \ X_j = (X_j^{(1)}, X_j^{(2)}) \ .$$

Then one can write

$$\tilde{A}_1(\zeta)X_j(\zeta')M_j(\zeta') + \tilde{A}_0(\zeta)X_j(\zeta') = 0 \ .$$

As in previous sections denote

$$X_{F1} = (X_1, X_2, \ldots, X_t), \ X_F = (X_0, X_{F1}), \ X = (X_F, X_\infty) \ .$$

Following the division of the eigenvalues $\kappa_j'$, $j = 1,2,\ldots,t$, into the groups I and II we relate the matrix $X_j$ to one of these groups. Then the matrix $X_{F1}$ is partitioned accordingly as $(X_I, X_{II})$. Having the partition $X_j = (X_j^{(1)}, X_j^{(2)})$ we obtain the corresponding partitions

$$X_I = (X_I^{(1)}, X_I^{(2)}) \quad \text{and} \quad X_{II} = (X_{II}^{(1)}, X_{II}^{(2)}) \ .$$

In a similar way define the matrices

$$M_{F1}' = \mathrm{diag}(M_1', M_2', \ldots, M_t'), \ M_{F1} = -I + rM_{F1}', \quad M_F = M_0 \oplus M_{F1}$$

and their partitions

$$M_{F1}' = (M_I' \oplus M_{II}') \text{ and } M_{F1} = (M_I \oplus M_{II}) \ .$$

As usual $T(\zeta') = (A_1(\zeta)X_F(\zeta'), A_0(\zeta)X_\infty(\zeta'))$ and the rows of the inverse matrix $X^{-1}(\zeta')$ are partitioned and denoted according to the columns of $X(\zeta')$. Defining $\tilde{B}_0(\zeta')$ and $\tilde{B}_1(\zeta')$ as in (7.42) we arrive at the identity

$$\tilde{L}(\kappa,\zeta)X(\zeta') = T(\zeta')(\tilde{B}_0(\zeta') + \kappa\tilde{B}_1(\zeta')) \,. \tag{9.23}$$

For $r \neq 0$ the matrix $\tilde{L}(\kappa,\zeta)$ is regular and therefore the matrices $X(\zeta')$ and $T(\zeta')$ are invertible. For $r = 0$ we have

$$\mathrm{Sp}(X_I^{(1)}(\zeta')) = F_1(-1)\,\mathrm{Im}\, P_I^{(1)}(\zeta')$$

and similar formulas hold for $X_I^{(2)}$, $X_{II}^{(1)}$ and $X_{II}^{(2)}$. If equalities (9.15) and (9.16) hold at the point $\zeta_0'$, the columns of the matrix $(X_0(\zeta_0'), X_\infty^{(1)}(\zeta_0'), X_I(\zeta_0'))$ are independent and form a basis of the space $F_1(-1)\mathbb{C}^n + \mathrm{Ker}\,\tilde{B}$. Similarly the columns of $(X_0(\zeta_0'), X_\infty^{(1)}(\zeta_0'), X_{II}(\zeta_0'))$ form a basis of the same space. Then we can represent $X_{II}(\zeta')$ for $\zeta' = (\xi', z', 0) \in \Omega(\zeta_0')$ as a linear combination

$$X_{II}(\zeta') = X_I(\zeta')C_I(\zeta') + X_0(\zeta')C_0(\zeta') + X_\infty^{(1)}(\zeta')C_\infty(\zeta') \,, \tag{9.24}$$

where the matrices $C_I(\zeta')$, $C_0(\zeta')$ and $C_\infty(\zeta')$ depend analytically on $\zeta'$.

Lemma 9.2. a) The matrices $\hat{T}^{-1}(\zeta') = r^2 T^{-1}(\zeta')$, $r(T^{-1}(\zeta'))_{F1}$ and $r(T^{-1}(\zeta'))_\infty^{(2)}$ are analytic in $\Omega(\zeta_0')$.

b) The last row of the matrix $(\hat{T}^{-1}(\zeta_0'))_\infty^{(1)}$ is non-zero.

Proof: Suppose first that $\mathrm{Re}\, z_0' > 0$ and equalities (9.15), (9.16) are satisfied at the point $\zeta_0'$. If $\mathrm{Re}\,\kappa_j' > 0$ let us define a function

$$\varphi_j(\kappa,\zeta') = |\kappa - M_0(\zeta')|\Big(\prod_i \Big|\frac{\kappa I - M_i(\zeta')}{\kappa+1+r}\Big|\Big)\Big/\Big(\frac{\kappa+1+r}{r}\Big)^2$$

$$= (-1+r\kappa')\Big(\prod_i \Big|\frac{\kappa' I - M_i'(\zeta')}{\kappa'+1}\Big|\Big)/(\kappa'+1)^2 = \psi_j(\kappa',\zeta')$$

where the product is taken over $1 \leq i \leq t$, $i \neq j$. For $r > 0$ the function

$\varphi_j(\kappa,\zeta')$ depends analytically on $\kappa$ in the unit disc $|\kappa| \leq 1$. The mapping $\kappa \to \kappa' = (\kappa+1)/r$ transforms the unit circle $|\kappa| = 1$ into the circle $|\kappa'-(1/r)| = 1/r$ in the half plane $\operatorname{Re} \kappa' \geq 0$. It is easy to verify that the integrals $\oint_{|\kappa|=1} |\psi_j(\kappa',\zeta')d\kappa'|$ are uniformly bounded for $\zeta' \in \Omega(\zeta_0')$ with $r > 0$. Let $\zeta' \in \Omega_R(\zeta_0')$. Stability of the Cauchy problem implies estimate (8.23) for any $\kappa$ with $|\kappa| = 1$. Multiplying the matrix $\tilde{L}^{-1}(\kappa,\zeta)$ by $\varphi_j(\kappa,\zeta')$ and integrating with respect to $\kappa$ around the unit circle $|\kappa| = 1$ we obtain

$$\|X_j(\zeta')\varphi_j(M_j(\zeta'),\zeta')(T^{-1}(\zeta'))_j\| \leq \frac{K}{|z|-1} \int_{|\kappa|=1} |\varphi_j(\kappa,\zeta')d\kappa|$$

$$= \frac{K}{|z|-1} \oint_{|\kappa|=1} |\psi_j(\kappa',\zeta')rd\kappa'| \leq \frac{K}{r} \quad .$$

Since $\varphi_j(M_j(\zeta'),\zeta') = \psi_j(M_j'(\zeta'),\zeta')$ is a *nonsingular* matrix and the columns of $X_j(\zeta_0')$ are independent, we get the estimate

$$\|(T^{-1}(\zeta'))_j\| \leq K/r \quad .$$

As in the proof of lemma 8.5 one can show that the matrix $T^{-1}(\zeta')$ has a singularity of the type $1/r^k$ and therefore the matrix $r(T^{-1}(\zeta'))_j$ is analytic in $\Omega(\zeta_0')$. Suppose now that (9.15) and (9.16) do not hold or $\operatorname{Re} z_0' = 0$. In $\Omega(\zeta_0')$ we can choose a point $\zeta_1' = (\xi_1',z_1',0)$ with real $\xi_1'$ and $\operatorname{Re} z_1' > 0$ such that (9.15) and (9.16) hold at $\zeta_1'$. Then the matrix $r(T^{-1}(\zeta'))$ is analytic in some neighbourhood $\Omega(\zeta_1') \subset \Omega(\zeta_0')$. Since the matrix $T^{-1}(\zeta')$ has in $\Omega(\zeta_0')$ a singularity of the type $1/r^k$, it follows that $r(T^{-1}(\zeta'))_j$ is analytic also in $\Omega(\zeta_0')$.

If Re $\kappa_j' < 0$, one should define $\varphi_j(\kappa,\zeta')$ as

$$\varphi_j(\kappa,\zeta') = |I/\kappa - M_\infty(\zeta')|\cdot|\kappa - M_0(\zeta')|\left(\prod_{i\neq j}\left|\frac{\kappa I - M_i(\zeta')}{\kappa+1-r}\right|\right)\Big/\left(\frac{\kappa+1-r}{r}\right)^2$$

so that the function $\kappa^{-1}\varphi_j(\kappa^{-1},\zeta')$ is analytic in the unit disc $|\kappa| \leq 1$. Then as before we get the estimate $\|(T^{-1}(\zeta')_j\| \leq K/r$ and the analyticity of $r(T^{-1}(\zeta'))_j$ follows. For $j = 0$ or $j = \infty$ the function $\varphi_j(\kappa,\zeta')$ is *defined* as in the proof of lemma 7.6. We arrive then at an estimate

$$\|(T^{-1}(\zeta'))_j\| \leq K/(|z|-1)$$

and the analyticity of $(\hat{T}^{-1}(\zeta'))_0$ and $(\hat{T}^{-1}(\zeta'))_\infty$ follows as *before.*

Let now $\zeta'\in\Omega(\zeta_0')$ with $r = 0$. We shall repeat the arguments used in lemma 8.5 in order to prove that $(\hat{T}^{-1}(\zeta'))_\infty^{(2)} = 0$. Let us fix in (9.23) a value of $\kappa$ different from the eigenvalues of $\tilde{B}_0(\zeta') + \kappa\tilde{B}_1(\zeta')$ for all $\zeta'\in\Omega(\zeta_0')$. If $v\in\mathrm{Im}\,\hat{T}^{-1}(\zeta')$, then also $v\in\mathrm{Ker}\,T(\zeta')$ and therefore

$$\tilde{L}(\kappa,\zeta_0)X(\zeta')(\tilde{B}_0(\zeta')+\kappa\tilde{B}_1(\zeta'))^{-1}v = 0 . \tag{9.24}$$

Denoting $u = (\tilde{B}_0(\zeta')+\kappa\tilde{B}_1(\zeta'))^{-1}v$ we have $X(\zeta')u\in\mathrm{Ker}\,\tilde{L}(\kappa,\zeta_0) \subset \mathrm{Ker}\,\tilde{B}$. The components of the vectors u and v are supposed to be partitioned according to the columns of $X(\zeta')$. Let us recall that the columns of *the matrix* $(X_j(\zeta'),X_{F1}(\zeta'),X_\infty^{(1)}(\zeta'))$ belong to the space $\mathrm{Ker}\,\tilde{B} + F_1(-1)\mathbb{C}^n$, and that the columns of $X_\infty^{(2)}(\zeta')$, which form a basis of the image of diag$(0,0,B,B,\ldots B)$, are independent of the above space. Therefore $u_\infty^{(2)} = 0$ and also $v_\infty^{(2)} = -M_\infty^{(2)}(\zeta')u_\infty^{(2)} = 0$. Hence $(\hat{T}^{-1}(\zeta'))_\infty^{(2)} = 0$ and the matrix $(rT^{-1}(\zeta'))_\infty^{(2)}$ is analytic in $\Omega(\zeta_0')$.

Let us now prove the second part of the lemma. Denote $\hat{X}^{-1}(\zeta') = r^2X^{-1}(\zeta')$. Using (9.23) we can write

$$\hat{X}^{-1}(\zeta') = (\tilde{B}_0(\zeta')+\kappa\tilde{B}_1(\zeta'))^{-1}\hat{T}^{-1}(\zeta')\tilde{L}(\kappa,\zeta') ,$$

where $\kappa$ is fixed as in (9.25). Since $\hat{T}^{-1}(\zeta')$ is analytic, it follows that $\hat{X}^{-1}(\zeta')$ too is analytic in $\Omega(\zeta_0')$. Let now $r = 0$. Since $(\hat{T}^{-1}(\zeta'))_{F1} = (\hat{T}^{-1}(\zeta'))_\infty^{(2)} = 0$, it follows from the block form of $\tilde{B}_0(\zeta') + \kappa\tilde{B}_1(\zeta')$ that also $(\hat{X}^{-1}(\zeta'))_{F1} = (\hat{X}^{-1}(\zeta'))_\infty^{(2)} = 0$. If $v\in\operatorname{Im}\hat{X}^{-1}(\zeta')$ then also $v\in\operatorname{Ker}X(\zeta')$, and since $v_{F1} = v_\infty^{(2)} = 0$, we obtain that $X_0(\zeta')v_0 + X_\infty^{(1)}(\zeta')v_\infty^{(1)} = 0$. But the columns of $(X_0(\zeta'), X_\infty^{(1)}(\zeta'))$ are independent, and hence $v_0 = v_\infty^{(1)} = 0$. So we have shown that $\hat{X}^{-1}(\zeta') = 0$ for $r = 0$ and therefore $rX^{-1}(\zeta')$ is analytic in $\Omega(\zeta_0')$. Let us represent the singular eigenvector $\tilde{\varphi}_0(\kappa,\pi)\in\operatorname{Ker}\tilde{B}$ as a linear combination

$$\tilde{\varphi}_0(\kappa,\pi) = X_0(\zeta')u_0(\zeta_0') + X_\infty^{(1)}(\zeta')u_\infty^{(1)}(\zeta_0') , \tag{9.26}$$

where $\zeta' = (\xi',z',0)$ and, hence, $X_0(\zeta')$, $X_\infty^{(1)}(\zeta')$ do not depend on $\xi'$ and $z'$. Since for different values of $\kappa$ the vectors $\tilde{\varphi}_0(\kappa,\pi)$ span the space $\operatorname{Ker}\tilde{B}$, we may assume that the last component of $u_\infty^{(1)}(\zeta_0')$ is non-zero. Let us define a vector $u(\zeta_0')\in\mathbb{C}^{mn}$ by completing $u_0(\zeta_0')$ and $u_\infty^{(1)}(\zeta_0')$ with zeros in the remaining components. Then for $\zeta' = (\xi',z',r)\in\Omega(\zeta_0')$ and $\xi = \pi-\xi'r$ we get

$$\tilde{\varphi}_0(\kappa,\xi) - X(\zeta')u(\zeta_0') = r\cdot\Delta\varphi(\zeta') ,$$

where $\Delta\varphi(\zeta')$ is analytic in $\Omega(\zeta_0')$. Then the vector function

$$\Delta u(\zeta') = rX^{-1}(\zeta')\Delta\varphi(\zeta')$$

is also analytic in $\Omega(\zeta_0')$ and defining $\tilde{u}(\zeta') = u(\zeta_0') + \Delta u(\zeta')$ we obtain

$$\tilde{\varphi}_0(\kappa,\xi) = X(\zeta')\tilde{u}(\zeta') .$$

Let us denote

$$(\tilde{B}_0(\zeta') + \kappa\tilde{F}_1(\zeta'))\tilde{u}(\zeta') = \tilde{v}(\zeta') .$$

Then

$$T(\zeta')\tilde{v}(\zeta') = \tilde{L}(\kappa,\zeta)\tilde{\varphi}_0(\kappa,\xi) = O(z-1) = O(r^2) .$$

Multiplying the last equality on the left by $T^{-1}(\zeta')$ we obtain that

$$\tilde{v}(\zeta') \in \operatorname{Im} \hat{T}^{-1}(\zeta') .$$

Let $\zeta' = (\xi',z',0)\in\Omega(\zeta_0')$. Since $(\hat{T}^{-1}(\zeta'))_{F1} = (\hat{T}^{-1}(\zeta'))_{\infty}^{(2)} = 0$, we conclude that also $\tilde{v}_{F1}(\zeta') = \tilde{v}_{\infty}^{(2)}(\zeta') = 0$ and therefore $\tilde{u}_{F1}(\zeta') = \tilde{u}_{\infty}^{(2)}(\zeta') = 0$. Then the uniqueness of representation (9.26) implies that $\tilde{u}_0(\zeta') = u_0(\zeta_0')$ and $\tilde{u}_{\infty}^{(1)}(\zeta') = u_{\infty}^{(1)}(\zeta_0')$. It follows now that

$$\tilde{v}_{\infty}^{(1)}(\zeta') = (I-\kappa M_{\infty}^{(1)}(\zeta'))u_{\infty}^{(1)}(\zeta_0') .$$

Since the last component of $u_{\infty}^{(1)}(\zeta_0')$ is non-zero and $M_{\infty}^{(1)}(\zeta')$ is a nilpotent Jordan cell, we conclude that the last component of $\tilde{v}_{\infty}^{(1)}(\zeta')$ and, therefore, the last row of $(\hat{T}^{-1}(\zeta_0'))_{\infty}^{(1)}$ are non-zero. The lemma is proved.

## 9.[illegible]. Block structure of the κ-matrix $\tilde{L}(\kappa,\zeta)$ in a neighbourhood $\Omega(\zeta_0')$ for $z_0'=0$.

Let $\kappa_j'$ be a root of the equation $p_0(-\xi_0',i\kappa',z_0' = 0)$ with multiplicity $q_j$. Then $\kappa_j'$ is a root of equation (9.8) with double multiplicity $2q_j$. The matrix $M'(\kappa_j',\zeta_0') = A\xi_0' + Bi\kappa_j'$ has a zero eigenvalue of some multiplicity $\rho > 1$. As in

Lemma 3.4 there is a non-singular matrix $D(\kappa',\zeta')$ analytic in $\Omega(\kappa_j')\times\Omega(\zeta_0')$, which provides a similarity transformation

$$(9.27)\qquad D^{-1}(\kappa',\zeta')C'(\kappa',\zeta')D(\kappa',\zeta') = \begin{bmatrix} N_0(\kappa',\zeta') & 0 \\ 0 & N_1(\kappa',\zeta') \end{bmatrix}$$

with

$$N_0(\kappa',\zeta') = \begin{bmatrix} 0 & 1 & & & \\ & & 1 & & \bigcirc \\ & \bigcirc & & \ddots & \\ & & & & 1 \\ 0 & e_1 & e_2 & \cdots & e_{\rho-1} \end{bmatrix},$$

where the coefficients $e_k = e_k(\kappa',\zeta')$ vanish at the point $(\kappa_j',\zeta_0')$ and the matrix $N_1(\kappa',\zeta')$ is non-singular in $\Omega(\kappa_j')\times\Omega(\zeta_0')$. We may assume that the first column of the matrix $D(\kappa',\zeta')$ is the singular eigenvector $\varphi_0(\alpha',\beta')$. Denote the second column of $D(\kappa',\zeta')$ by $\varphi_1(\kappa',\zeta')$. It is obvious that the kernel of the matrix $(C'(\kappa_j',\zeta_0'))^2$ is two-dimensional and $\varphi_1(\kappa_j',\zeta_0')$ is the second eigenvector of this matrix corresponding to the zero eigenvalue. Multiplying the matrix $\ell(N_0(\kappa',\zeta'),\kappa',\zeta')$ (the function $\ell$ is defined in 9.3) on the left by the matrix $E_1$ as in Lemma 3.4 and then by $E_2 = \mathrm{diag}(-(z'\kappa)^{-1},2,2,\dots,2)$ we obtain

$$E_2E_1\ell(N_0,\kappa',\zeta') = \begin{bmatrix} e_1 & e_2 & e_3 & \cdot & \cdot & e_{\rho-1}-1 \\ 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 1 & & \bigcirc \\ \vdots & & & & \ddots & \\ \vdots & & & & & 1 \\ 0 & e_1 & e_2 & \cdot & \cdot & e_{\rho-1} \end{bmatrix} + O(r) + O(z').$$

where the first column of the matrix $O(r)$ is zero. Multiplying the matrix thus obtained on the left by

$$E_3 = \begin{bmatrix} 1 & -e_3 & -e_4 & \cdot & \cdot & -e_{\rho-1} & 1 & 0 \\ 0 & -e_2 & -e_3 & \cdot & \cdot & \cdot & -e_{\rho-1} & 1 \\ 0 & 1 & 0 & \cdot & \cdot & \cdot & & \text{\Large 0} \\ & \text{\Large 0} & & & & & 1 & 0 \end{bmatrix}$$

we get

$$E_3E_2E_1\cdot \ell(N_0,\kappa',\zeta') = \begin{bmatrix} e_1 & e_2 & 0 & \text{\Large 0} \\ 0 & e_1 & 0 & \\ 0 & 0 & 1 & \\ & \text{\Large 0} & & \cdot \; 1 \end{bmatrix} + O(r) + O(z'),$$

where again the first column of the matrix $O(r)$ is zero. Let us denote the resulting matrix $E_3E_2E_1\cdot\ell(N_0,\kappa',\zeta')$ as $N_0'(\kappa',\zeta')$. Replace the operator $Q_j(\zeta')$ in (9.12) by a new one denoted by the same letter

$$(9.28)\quad Q_j(\zeta)\varphi = (2\pi i)^{-1}\oint_{\Gamma_j'} F_1(\kappa)D(\kappa',\zeta')[N_0'(\kappa',\zeta')\oplus\ell(N_1(\kappa',\zeta'),\kappa',\zeta')]^{-1}\varphi(\kappa')d\kappa'$$

$$= (2\pi i)^{-1}\oint_{\Gamma_j'} F_1(\kappa)D(\kappa',\zeta')[(N_0'(\kappa',\zeta'))^{-1}\oplus O_{n-\rho}]\varphi(\kappa')d\kappa' .$$

For $rz' \neq 0$ the images of the operators $Q_j(\zeta')$ in (9.28) and (9.12) coincide and have the dimension $2q_j$. The new operator $Q_j(\zeta')$ depends analytically on $\zeta'\in\Omega(\zeta_0')$. Also note that the expressions in the above two integrals become analytic in $\Omega(\kappa_j')\times\Omega(\zeta_0')$ when multiplied on the left by $\tilde{L}(\kappa,\zeta)$. As in lemma 3.4 we can write

$$e_1(\kappa',\zeta_0') = (\kappa'-\kappa_j')^{q_j} f_1(\kappa') \quad\text{and}\quad e_2(\kappa',\zeta_0') = (\kappa'-\kappa_j')^{r_j} f_2(\kappa'),$$

where $f_1(\kappa')$ and $f_2(\kappa')$ are invertible in $\Omega(\kappa'_j)$ and $r_j \geq 1$ is an integer. For $\zeta' = \zeta'_0$ the operator $Q_j(\zeta')$ in (9.28) may be written as

$$(9.29)\quad Q_j(\zeta'_0)\varphi = (2\pi i)^{-1}F_1(-1)\oint_{\Gamma'_j} D(\kappa',\zeta'_0)\left[\begin{pmatrix} e_1(\kappa',\zeta'_0) & e_2(\kappa',\zeta'_0) \\ 0 & e_1(\kappa',\zeta'_0)\end{pmatrix}^{-1} \oplus 0_{n-2}\right]\varphi(\kappa')d\kappa'.$$

If $q_j = 1$, the image of $Q_j(\zeta'_0)$ is spanned by two linearly independent vectors $F_1(-1)\varphi_0(-\xi'_0,i\kappa'_j)$ and $F_1(-1)\varphi_1(\kappa'_j,\zeta'_0)$ and has therefore the dimension $2q_j$. In the case of $q_j > 1$ we should make an additional

Assumption 9.2. The image of the operator $Q_j(\zeta'_0)$ in (9.29) has the dimension $2q_j$ for $j = 1,2,\ldots,t$.

Let us note that if the order n of the matrices A and B is equal to 3, then $t = 2$ and $q_1 = q_2 = 1$ so that assumption 9.2 is fulfilled.

Lemma 9.3. a) There exists matrix valued function $X_j(\zeta') = (X_j^{(1)}(\zeta'),X_j^{(2)}(\zeta'))$, $j = 1,2,\ldots,t$, analytic in $\Omega(\zeta'_0)$, whose columns form a basis of the image of $Q_j(\zeta')$ in (9.28) for any $\zeta'\in\Omega(\zeta'_0)$.

b) For $z' = 0$ the columns of $X_j^{(1)}(\zeta')$ belong to Ker $\tilde{P}(\xi)$ (where $\xi=\pi-\xi'r$) and $X_j^{(1)}(\zeta'_0) = F_1(-1)Y'_j(\zeta'_0)$, where the columns of $Y'_j(\zeta'_0)$ form a singular Jordan chain of length $q_j$ corresponding to the eigenvalue $\kappa' = \kappa'_j$ of the singular $\lambda$-matrix $- A\xi'_0 + Bi\kappa'$.

c) There is a matrix valued function $M'_j(\zeta')$ of order $2q_j\times 2q_j$ analytic in $\Omega(\zeta'_0)$ such that the identity

$$(9.30)\quad \tilde{A}_1(\zeta)X_j(\zeta')M_j(\zeta') + A \text{ [illegible]} \qquad \text{with } M_j(\zeta') = -I+rM'_j(\zeta')$$

holds for any $\zeta'\in\Omega(\zeta'_0)$ [illegible] matrix $M'_j(\zeta'_0)$ has the only eigenvalue $\kappa'_j$.

ii) The matrix $M_j'(\zeta')$ is partitioned according to the partition of $X_j(\zeta')$:

$$M_j'(\zeta') = \begin{bmatrix} M_{j11}'(\zeta') & M_{j12}'(\zeta') \\ M_{j21}'(\zeta') & M_{j22}'(\zeta') \end{bmatrix}, \tag{9.31}$$

where $M_{j21}'(\zeta') = 0$ for $z' = 0$.

Proof: Since $\dim \operatorname{Im} Q_j(\zeta') = 2q_j$ for $\zeta' \in \Omega(\zeta_0)$ with $rz' \neq 0$ and for $\zeta' = \zeta_0'$, the dimension of $\operatorname{Im} Q_j(\zeta')$ is constant for all points $\zeta'$ of sufficiently small neighbourhood $\Omega(\zeta_0')$ and the image of $Q_j(\zeta')$ depends analytically on $\zeta' \in \Omega(\zeta_0')$. Denote by $Q_j^{(1)}(\zeta')$ the restriction of $Q_j(\zeta')$ on the space of vector functions $\varphi(\kappa') = (\varphi^{(1)}(\kappa'),0,\dots,0)'$, where $\varphi^{(1)}(\kappa')$ is a scalar function. For $z' = 0$ we have

$$Q_j^{(1)}(\zeta')\varphi = (2\pi i)^{-1} \oint_{\Gamma_j'} F_1(\kappa)\varphi_0(\alpha',\beta')e_1^{-1}(\kappa',\zeta')\varphi^{(1)}(\kappa')d\kappa' .$$

For $r \neq 0$

$$F_1(\kappa)\varphi_0(\alpha',\beta') \sim F_1(\kappa)\varphi_0(\alpha,\beta) = \tilde{\varphi}_0(\kappa,\xi) \quad \in \operatorname{Ker} \tilde{P}(\xi)$$

and for $r = 0$

$$F_1(\kappa)\varphi_0(\alpha',\beta') = F_1(-1)\varphi_0(\alpha',\beta') \in F_1(-1)V_0 \subset \operatorname{Ker} \tilde{P}(\pi).$$

Hence for $z' = 0$ the image of $Q_j^{(1)}(\zeta')$ belongs to $\operatorname{Ker} \tilde{P}(\xi)$. Substituting for $\varphi^{(1)}(\kappa')$ respectively the functions $(\kappa'-\kappa_j')^{q_j-1} \cdot f_1(\kappa')$, $(\kappa'-\kappa_j')^{q_j-2} \cdot f_1(\kappa')$, $\dots, f_1(\kappa')$ and applying the operator $Q_j^{(1)}(\zeta')$ we obtain the $q_j$ columns of the matrix $X_j^{(1)}(\zeta')$. Since $e_1(\kappa',\zeta_0') = (\kappa'-\kappa_j')^{q_j} f_1(\kappa')$, it is clear that $X_j^{(1)}(\zeta_0') = F_1(-1)X_j'(\zeta_0')$ where the columns of $X_j'(\zeta_0')$ form a Jordan chain of length $q_j$ generated by the singular root function $\varphi_0(-\xi_0',i\kappa')$ at the point $\kappa' = \kappa_j'$. Therefore the columns of $X_j'(\zeta_0')$ and, hence, those of $X_j^{(1)}(\zeta_0')$ are independent. If $\Omega(\zeta_0')$ is small enough, the columns of $X_j^{(1)}(\zeta')$ will be independent for any $\zeta'\in\Omega(\zeta_0')$ and form a basis of Im $Q_j^{(1)}(\zeta')$ for $z' = 0$. One can add to these columns other $q_j$ ones, which depend analytically on $\zeta'$, in order to form a basis of the $2q_j$-dimensional space Im $Q_j(\zeta')$. We shall denote the $nm\text{x}2q_j$ matrix thus obtained by $X_j(\zeta')$ and partition it as $(X_j^{(1)}(\zeta'), X_j^{(2)}(\zeta'))$. It may be assumed that $X_j(\zeta') = Q_j(\zeta')(\Psi(\kappa'))$, where $\Psi(\kappa') = (\Psi^{(1)}(\kappa'),\Psi^{(2)}(\kappa'))$ is an $n\text{x}2q_j$ matrix analytic in $\Omega(\kappa_j')$. The matrix $M_j'(\zeta')$ is then defined by the equality $Q_j(\zeta')(\kappa'\Psi(\kappa')) = X_j(\zeta')M_j'(\zeta')$. Now, as in the case $z_0' \neq 0$, one can show that identity (9.30) is satisfied and that the matrix $M_j'(\zeta_0')$ has the only eigenvalue $\kappa_j'$. Since $Q_j(\zeta')(\kappa'\Psi^{(2)}(\kappa'))=$ $= Q_j^{(1)}(\zeta')(\kappa'\Psi^{(2)}(\kappa'))$ and for $z' = 0$ the columns of $X_j^{(1)}(\zeta')$ form a basis of the space Im $Q_j^{(1)}(\zeta')$, it follows that $M_{j21}'(\zeta') = 0$ for $z' = 0$.

Using the matrices $X_0(\zeta') = X_0(\zeta)$ and $X_\infty(\zeta') = X_\infty(\zeta)$ defined in lemma 9.1 we build the whole nmxnm matrix

$$X(\zeta') = (X_0(\zeta'),X_1(\zeta'),\ldots,X_t(\zeta'),X_\infty(\zeta'))$$

and partition it as in the case $z_0' \neq 0$. We additionally partition it as

$$X = (X^{(1)},X^{(2)}),\quad X^{(1)} = (X_0,X_{F1}^{(1)},\ X_\infty^{(1)}),\ X^{(2)} = (X_{F1}^{(2)},X_\infty^{(2)}),$$

where

$$X_{F1}^{(1)} = (X_1^{(1)},\ldots,X_t^{(1)}) \text{ and } X_{F1}^{(2)} = (X_1^{(2)},\ldots,X_t^{(2)}) \ .$$

The matrices $T(\zeta')$, $T^{-1}(\zeta')$ and $M_F(\zeta')$ are determined as in the previous subsection and partitioned according to $X(\zeta')$. Defining $\tilde{R}_0(\zeta')$ and $\tilde{R}_1(\zeta')$ as in

(9.22) we arrive at identity (9.23). For $rz' \neq 0$ the κ-matrix $\tilde{L}(\kappa,\zeta)$ is regular and the matrices $X(\zeta')$ and $T(\zeta')$ are invertible.

*Lemma 9.4.* a) For $z' = 0$ the columns of $X^{(1)}(\zeta')$ belong to the space Ker $\tilde{P}(\xi)$, where $\xi = \pi-\xi'r$, and the columns of $(X_0(\zeta'),X_I^{(1)}(\zeta'), X_\infty^{(1)}(\zeta'))$ as well as those of $(X_0(\zeta'),X_{II}^{(1)}(\zeta'), X_\infty^{(1)}(\zeta'))$ form a basis of Ker $\tilde{P}(\xi)$.
b) For $\zeta' = (\xi',0,r)\in\Omega(\zeta'_0)$ with real $r \neq 0$ and real $\xi'$ the columns of $X^{(2)}(\zeta')$ are independent of the space Ker $\tilde{P}(\xi)$.

*Proof:* The first part of the lemma is proved exactly as part b) of lemma 8.6. Let us go on to the second part of the lemma. Fix a point $\zeta'_1 = (\xi'_1,0,r_1)\in\Omega(\zeta'_0)$ with real $\xi'_1$ and $r_1 \neq 0$, and denote $\xi_1 = \pi-\xi'_1r_1 \neq \pi$, $\zeta_1 = (\xi_1,z=1)$. In subsection 7.1 we have investigated the block structure of the κ-matrix $\tilde{L}(\kappa,\zeta)$ in some neighbourhood $\Omega(\zeta_1)$ of the point $\zeta_1$. To avoid confusion we denote the corresponding matrix $X(\zeta)$ by $Y(\zeta)$. We shall consider only the points $\zeta = (\xi_1,z)\in\Omega(\zeta_1)$ with z sufficiently small so that the corresponding point $\zeta' = (\xi'_1,z',r_1)$ with $z' = (z-1)/r_1^2$ belongs to $\Omega(\zeta'_0)$. Denote by $\Gamma_j$, $j = 1,2,\dots,t$, the constant contour $-1 + r_1\Gamma'_j$. The [illegible] eigenvalues of the κ-matrix $\tilde{L}(\kappa,\zeta)$ near the point $\kappa = -1$ are subdivided into t groups surrounded by the contours $\Gamma_j$. These eigenvalues do not cross the contours $\Gamma_j$ since the corresponding eigenvalues $\kappa'$ of the $\kappa'$-matrix $L'(\kappa',\zeta')$, where $\zeta' = (\xi'_1,z',r_1)\in\Omega(\zeta'_0)$, do not cross the contours $\Gamma'_j$. The columns of $Y(\zeta)$ are partitioned as $Y = (Y_0,Y_{F1},Y_\infty)$ and also $Y = (Y^{(1)},Y^{(2)})$ as in (7.38). According to the above subdivision of the eigenvalues κ, we partition $Y_{F1} = (Y_1,Y_2,\dots,Y_t)$. Let us note that any matrix $X_j(\zeta)$ in the old notation of (7.37) is included as a whole in one of the matrices $Y_j(\zeta)$. The matrices $Y_j$ are also partitioned as $(Y_j^{(1)},Y_j^{(2)})$ so that the columns of $Y_j^{(1)}(\zeta_1)$ belong to Ker $\tilde{P}(\xi_1)$ and those of $Y^{(2)}(\zeta_1) = (Y_1^{(2)}(\zeta_1),\dots, Y_t^{(2)}(\zeta_1),Y_\infty^{(2)}(\zeta_1))$ are independent of Ker $\tilde{P}(\xi_1)$. For $\zeta' = (\xi'_1,z' \neq 0, r_1)\in\Omega(\zeta'_0)$ and corresponding $\zeta = (\xi_1,z)\in\Omega(\zeta_1)$, the spaces spanned by the columns of $X_j(\zeta')$

and $Y_j(\zeta)$ coincide since they both are equal to the image of the projector $P_j(\zeta')$ in [illegible]. Then from continuity considerations we obtain also $\operatorname{Sp} X_j(\zeta_1') = \operatorname{Sp} Y_j(\zeta_1)$. The matrix $Y_j^{(1)}$ has as many columns as the number of $\kappa$-roots of the equation $p_0(\alpha,\beta,\kappa) = 0$ surrounded by the contour $\Gamma_j$. Here the functions $\alpha$ and $\beta$ are given by (6.10) and correspond to $\zeta = \zeta_1$. To any such root $\kappa$ corresponds a root $\kappa' = (\kappa+1)/r$ of the equation $p_0(\alpha',\beta',\kappa') = 0$ surrounded by the contour $\Gamma_j'$, where the functions $\alpha'$ and $\beta'$ are given by (9.3) with $\zeta' = \zeta_1'$. Since the number of such roots $\kappa'$ is equal to $q_j$, it follows that the matrices $X_j^{(1)}(\zeta_1')$ and $Y_j^{(1)}(\zeta_1)$ have the same order $nm \times q_j$. Since

$$\operatorname{Sp} Y_j^{(1)}(\zeta_1) = \operatorname{Sp} Y_j(\zeta_1) \cap \operatorname{Ker} \tilde{F}(\xi_1) = \operatorname{Sp} X_j(\zeta_1') \cap \operatorname{Ker} \tilde{F}(\xi_1)$$

and the columns of $X_j^{(1)}(\zeta_1')$ are independent and belong to $\operatorname{Ker} \tilde{F}(\xi_1)$, we conclude that $\operatorname{Sp} X_j^{(1)}(\zeta_1') = \operatorname{Sp} Y_j^{(1)}(\zeta_1)$. Hence there is a relation

$$(X_j^{(1)}(\zeta_1'),\ X_j^{(2)}(\zeta_1')) = (Y_j^{(1)}(\zeta_1), Y_j^{(2)}(\zeta_1)) \begin{bmatrix} c_{j11} & c_{j12} \\ 0 & c_{j22} \end{bmatrix}, \quad j = 1,2,\dots,t,$$

where the matrices $c_{j11}$ and $c_{j22}$ are non-singular. For $j = 0$ and $j = \infty$ the matrices $X_j(\zeta_1')$ and $Y_j(\zeta_1)$ coincide. Since the columns of $Y^{(2)}(\zeta_1)$ are independent of $\operatorname{Ker} \tilde{F}(\xi_1)$, the columns of $X^{(2)}(\zeta_1')$ have the same feature. The lemma is proved.

Analogously to lemma 9.2 we have the following

Lemma 9.2'. a) The matrices $\tilde{T}^{-1}(\zeta') = r' S' T^{-1}(\zeta')$, $rS'(T^{-1}(\zeta'))^{(1)}_{F'}$ and $(T^{-1}(\zeta'))^{(2)}$ are analytic in $\Omega(\zeta_1')$.

b) The last row of the matrix $(T^{-1}(\zeta'))^{(1)}_{n}$ is non-zero.

Proof: Using the independence of the columns of $X_j(\zeta')$ one obtains as in lemma 9.2 the estimates

$$\|\hat{T}^{-1}(\zeta')\| \leq \frac{K|r^2z'|}{|z|-1} \quad \text{and} \quad \|rz'(T^{-1}(\zeta'))_{F1}\| \leq \frac{K|r^2z'|}{|z|-1} .$$

Then repeating the corresponding arguments used in part a) of *lemma* 8.7 we prove the analyticity of $\hat{T}^{-1}(\zeta')$ and $rz'(T^{-1}(\zeta'))_{F1}$. Let $\zeta' = (\xi',0,r)\in\Omega(\zeta'_0)$ with real $r \neq 0$ and [illegible]'. Since $\text{Im}\ \hat{T}^{-1}(\zeta') \subset \text{Ker}\ T(\zeta')$ and the columns of $X^{(2)}(\zeta')$ are independent of $\text{Ker}\ \tilde{P}(\xi) = \tilde{V}_0(\xi)$, we obtain as in lemma 7.6 that $(T^{-1}(\zeta'))^{(2)} = 0$ and hence the matrices $r(T^{-1}(\zeta'))^{(2)}_{F1}$ and $r^2(T^{-1}(\zeta'))^{(2)}$ are analytic in $\Omega(\zeta'_0)$. As in *lemma* 9.2 one can show that $(\hat{T}^{-1}(\zeta'))^{(2)}_\infty = 0$ for $r = 0$ and $z' \neq 0$. Therefore also the matrix $r(T^{-1}(\zeta'))^{(2)}_\infty$ is analytic in $\Omega(\zeta'_0)$.

Let us now prove the second part of the lemma. As in lemma 9.2 it may be shown that the matrix $rz'X^{-1}(\zeta')$ is analytic in $\Omega(\zeta'_0)$. We shall apply now arguments already used in part (b) of lemma 8.7. Let us fix $\kappa$ different from the [illegible] of $\tilde{P}_0(\zeta') + r\tilde{P}_1(\zeta')$ for all $\zeta'\in\Omega(\zeta'_0)$ and represent the vector [illegible] by

$$\tilde{\varphi}_0(\kappa,\pi) = X_0(\zeta')u_0(\zeta'_0) + X^{(1)}_\infty(\zeta')u^{(1)}_\infty(\zeta'_0),$$

[illegible] $\zeta' = (\xi',z',r)\in\Omega(\zeta'_0)$. As in (9.2[illegible]) we may assume that the last component [illegible] is non-zero. Let us define a vector $u(\zeta'_0)\in\mathbb{C}^{mn}$ by completing $u_0(\zeta'_0)$ [illegible] with zeros in the remaining components. Then for $\zeta' = (\xi',z',r)\in$ [illegible] and $\xi = \pi-\xi'r$ we get

$$\tilde{\varphi}_0(\kappa,\xi) - X(\zeta')u(\zeta'_0) = r\cdot\Delta\varphi(\zeta'),$$

where $\Delta\varphi(\zeta')$ is analytic in $\Omega(\zeta_0')$ and for $z' = 0$, $\Delta\varphi(\zeta')\in\mathrm{Ker}\,\tilde{P}(\xi)$. Since the columns of $(X_0(\zeta'),X_I^{(1)}(\zeta'),X_\infty^{(1)}(\zeta'))$ form for $z' = 0$ a basis of Ker $\tilde{P}(\xi)$, there exists a vector function $\Delta u(\zeta')$ analytic in $\Omega(\zeta_0')$ such that

$$\Delta u^{(2)}(\zeta') = 0 \text{ and } \Delta\varphi(\zeta') - X^{(1)}(\zeta')\Delta u^{(1)}(\zeta') = z'\Delta\psi(\zeta') ,$$

where $\Delta\psi(\zeta')$ is analytic in $\Omega(\zeta_0')$. Then the function $\Delta\tilde{u}(\zeta') = X^{-1}(\zeta')(rz'\Delta\psi(\zeta'))$ is analytic in $\Omega(\zeta')$, and defining $\tilde{u}(\zeta') = u(\zeta_0')+ r\Delta u(\zeta') +\Delta\tilde{u}(\zeta')$ we have $\tilde{\varphi}_0(\kappa,\xi) = X(\zeta')\tilde{u}(\zeta')$. Introducing $v(\zeta') = (\tilde{E}_0(\zeta') + \kappa\tilde{B}_1(\zeta'))\tilde{u}(\zeta')$ we get from [illegible] that

$$T(\zeta')v(\zeta') = \tilde{L}(\kappa,\zeta)X(\zeta')\tilde{u}(\zeta') = \tilde{L}(\kappa,\zeta)\tilde{\varphi}_0(\kappa,\xi)=O(z-1) = O(r^2z') .$$

Then $v(\zeta') = T^{-1}(\zeta')O(r^2z')$ and since the matrix $\hat{T}^{-1}(\zeta') = r^2z'T^{-1}(\zeta')$ is analytic, we obtain that $v(\zeta')\subset\mathrm{Im}\,\hat{T}^{-1}(\zeta')$ for $rz' \neq 0$ and therefore also $v(\zeta_0')\in\mathrm{Im}\,\hat{T}^{-1}(\zeta_0')$. It remains only to show that the last component of $v_\infty^{(1)}(\zeta_0')$ is non-zero. According to part a) of this lemma all rows of $\hat{T}^{-1}(\zeta_0')$ except $(\hat{T}^{-1}(\zeta_0'))_0$ and $(\hat{T}^{-1}(\zeta_0'))_\infty^{(1)}$ vanish. Therefore $v_{FI}(\zeta_0')= v_\infty^{(2)}(\zeta_0') = 0$ and, because of the block form of the matrix $\tilde{E}_0(\zeta_0') + \kappa\tilde{B}_1(\zeta_0')$, also $\tilde{u}_{FI}(\zeta_0') = \tilde{u}_\infty^{(2)}(\zeta_0') = 0$. Therefore $\tilde{\varphi}_0(\kappa,\xi) = X_0(\zeta_0')\tilde{u}_0(\zeta_0') + X_\infty^{(1)}(\zeta_0')\tilde{u}_\infty^{(1)}(\zeta_0')$ and the uniqueness of representation [illegible] implies that $\tilde{u}(\zeta_0') = u(\zeta_0')$, and hence the last component of $\tilde{u}_\infty^{(1)}(\zeta_0')$ is non-zero. Then the last component of $v_\infty^{(1)}(\zeta_0') = (I-\kappa M_\infty^{(1)}(\zeta_0'))\tilde{u}_\infty^{(1)}(\zeta_0')$ is also non-zero and the lemma is proved.

In order to prove theorems 5.1-5.3 locally in a neighbourhood $\Omega(\zeta_0')$ of a point $\zeta_0'$ with $z_0' = 0$ we shall need in the next subsection the following

[illegible] The columns of the matrix $(X_0(\zeta_0'),X_I(\zeta_0'),X_\infty^{(1)}(\zeta_0'))$ are independent.

Since all the n+m-1 columns of the matrix $(X_0(\zeta'), X_I(\zeta'), X_\infty^{(1)}(\zeta'))$ belong for r = 0 to the n+m-1 dimensional space Ker $\tilde{B} + F_1(-1)\mathbb{C}^n$, they also form a basis of this space if $\zeta'$ is close enough to $\zeta'_0$. Therefore representation (9.24) is still valid for the points $\zeta' = (\xi', z', 0) \in \Omega(\zeta'_0)$.

In the case n = 3 it may be shown that assumption 9.3 is fulfilled. We can also prove lemmas 9.3-9.5 without using assumption 9.2. Then only part a) of lemma 9.3 should be reformulated so that the columns of $X_j(\zeta')$ span the space Im $Q_j(\zeta')$ for any $\zeta' \in \Omega(\zeta'_0)$ and are independent for $r \neq 0$. Thus, assumptions 9.1-9.3 are connected with the boundary value problem and its stability and not with the block structure of the κ-matrix $\tilde{L}(\kappa, \zeta)$.

9.[illegible]. Proof of theorems 5.1-5.3 in the neighbourhood $\Omega(\zeta'_0)$.

We consider first the case $z'_0 \neq 0$. The operator P in estimate (6.9) is defined as $P = \tilde{B} = \mathrm{diag}(B, B, \ldots, B)$. Theorem 5.3 is formulated now in the following form

Sufficiency: If (UKC) is satisfied in $\Omega(\zeta'_0)$ and dim $\tilde{S}(\pi,1)$ Ker $\tilde{B} = 1$, estimate (6.9) holds in $\Omega(\zeta'_0)$ with $|z_0| = 1$.

Necessity: If estimate (6.9) holds in $\Omega(\zeta'_0)$ with $|z_0| = 1+\alpha_0\Delta x$, where [illegible], and $\tilde{S}(\pi,1)\, X(\zeta'_0) \neq 0$, then (UKC) is satisfied in $\Omega(\zeta'_0)$ and dim $\tilde{S}(\pi,1)$Ker $\tilde{B} = 1$.

Theorem 5.2 is replaced by the stronger theorem 5.3 and theorem 5.1 is formulated locally as in subsection 8.3. Applying to equation (7.44) the usual transformation $v(x) = X^{-1}(\zeta')u(x)$, $G(x) = T^{-1}(\zeta')F(x)$ we arrive at equations (7.45) where boundary condition (7.45) (c) should be written in the form ([illegible]). The diagonal blocks of the symmetrizer $R(\zeta')$ are defined as in subsection [illegible]. Namely,

$$R_0(\zeta') = -erI, \quad R_\infty(\zeta') = R_\infty^{(1)}(\zeta') \oplus R_\infty^{(2)}(\zeta') = (rI) \oplus I$$

$R_j(\zeta') = -cI$ when $\mathrm{Re}\,\kappa'_j > 0$ and $R_j(\zeta') = I$ when $\mathrm{Re}\,\kappa'_j < 0$, $j = 1,2,\dots,t$. Since

$$\mathrm{Re}\, R_j(\zeta')M'_j(\zeta') \leq -\delta I \text{ and } M_j(\zeta') = -I+rM'_j(\zeta'),$$

it follows that

$$M^*_j(\zeta')R_j(\zeta')M_j(\zeta') - R_j(\zeta') \geq \delta r\, I$$

for sufficiently small $r > 0$. Then the symmetrizers $R_F(\zeta')$ and $R_\infty(\zeta')$ satisfy for any $\zeta' \in \Omega_R(\zeta'_0)$ the conditions

$$(9.33)\quad M^*_F(\zeta')R_F(\zeta')M_F(\zeta') - R_F(\zeta') \geq \delta r\, I,\quad R_\infty(\zeta')-M^*_\infty(\zeta')R_\infty(\zeta')M_\infty(\zeta') \geq \delta r\, I$$

$$(9.34)\quad v^*_{F1}\, R_{F1}(\zeta')v_{F1} \geq -c|v_I|^2+|v_{II}|^2,\quad v^*_0R_0(\zeta')v_0 \geq -cr|v_0|^2,\quad v^*_\infty R_\infty(\zeta')v_\infty \geq r|v_\infty|^2 .$$

Applying to equations (7.45)(A),(B) the generalized energy method as in subsection 7.2 we arrive at an estimate

$$(9.35)\quad \delta r\|v\|^2 + [|v_{II}(0)|^2+|v_\infty^{(2)}(0)|^2+r|v_\infty^{(1)}(0)|^2-c(|v_I(0)|^2+r|v_0(0)|^2)]\Delta x$$
$$\leq K\|R(\zeta')\hat{h}\|^2/r .$$

Let us note that $\|u\|^2 = \|X(\zeta')v\|^2 \leq K\|v\|^2$. Since the norm of the matrices $r(T^{-1}(\zeta'))_0$, $r(T^{-1}(\zeta'))_\infty^{(1)}$, $(T^{-1}(\zeta'))_{F1}$ and $(T^{-1}(\zeta'))_\infty^{(2)}$ is bounded by $K/r$, we get an estimate

$$\|R(\zeta')\hat{h}\|^2 \leq K\|\hat{h}\|^2/r^2 .$$

As in subsection 8.3 one can show that in $\Omega(\zeta'_0)$ the condition (UKC) is equivalent to the condition $\det \tilde{C}(\zeta_0)(X(\zeta'),X(\zeta'_0)) \neq 0$ (provided $\Omega(\zeta'_0)$ is sufficiently small). The proof is simple since the matrix $M_{F1}(\zeta')$ is partitioned on blocks $M_I(\zeta')$ and $M_{II}(\zeta')$ with eigenvalues belonging for $r > 0$ to the domains $|\kappa| < 1$ and $|\kappa| > 1$ respectively. We use also the fact (based on assumption 9.1) that the columns of $(X_0(\zeta_0'), X_I(\zeta_0'))$ are independent.

Consider the boundary condition (8.51). If (UKC) is fulfilled, we get an estimate

$$|v_0(0)|^2 + |v_I(0)|^2 \le K(|v_{II}(0)|^2 + |v_\infty(0)|^2 + |g|^2) . \tag{9.36}$$

If additionally dim $\tilde{S}(\zeta_0)$ Ker $\tilde{B} = 1$, we get as in subsection 8.3 the estimate (8.58) and rewrite it here:

$$|v_I(0)|^2 + r|v_0(0)|^2 \le K(|v_{II}(0)|^2 + |v_\infty^{(2)}(0)|^2 + r|v_\infty^{(1)}(0)|^2 + |g|^2) . \tag{9.37}$$

Choosing then the constant c in (9.35) sufficiently small we obtain from (9.36) and (9.37) an estimate

$$r\|u\|^2 + (|v_{FI}(0)|^2 + |v_\infty^{(2)}(0)|^2 + r|v_0(0)|^2 + r|v_\infty^{(1)}(0)|^2)\Delta x$$
$$\le K\left(\frac{\|F\|^2}{r^3} + |g|^2 \Delta x\right) . \tag{9.38}$$

Assuming that $v_0(0)$ and $v_I(0)$ are linear functions of $g$, $v_{II}(0)$ and $v_\infty(0)$ given by equation (8.51), we may consider also the vector $\tilde{B}u(0) = \tilde{R}X(\zeta')v(0)$ as a linear function of $g$, $v_{II}(0)$ and $v_\infty(0)$ with coefficients analytic in $U(\zeta_0')$. We claim that there is an estimate

$$|\tilde{B}u(0)|^2 \le K(|g|^2 + |v_\infty^{(2)}(0)|^2 + |rv_\infty^{(1)}(0)|^2 + |rv_{II}(0)|^2) . \tag{9.39}$$

It is enough to show that $\tilde{B}u(0) = 0$ if $r = g = v_\infty^{(2)}(0) = 0$. Then $u(0)$ is a linear combination of the columns of $(X_0(\zeta'), X_{FI}(\zeta'), X_\infty^{(1)}(\zeta'))$ and, according to (9.24), $u(0)$ may be written as a linear combination

$$u(0) = X_0(\zeta')w_0 + X_I(\zeta')w_I + X_\infty^{(1)}(\zeta')w_\infty^{(1)} .$$

Since $\tilde{S}(\zeta_0)u(0) = g = 0$, we obtain that

$$\tilde{S}(\zeta_0)X_I(\zeta')w_I = -\tilde{S}(\zeta_0)(X_0(\zeta')w_0 + X_\infty^{(1)}(\zeta')w_\infty^{(1)}) .$$

For $r = 0$ the columns of $X_0(\zeta') = X_0(\zeta_0)$ and $X_\infty^{(1)}(\zeta') = X_\infty^{(1)}(\zeta_0)$ belong to the space Ker $\tilde{B}$ and according to condition 5.1 the right hand side of the last equality is proportional to the vector $\tilde{S}(\zeta_0)X_0(\zeta')$. Since det $\tilde{S}(\zeta_0)(X_0(\zeta'),X_I(\zeta'))\neq 0$, it follows that $w_I = 0$. *Hence* the vector $u(0)$ belongs to Ker $\tilde{B}$ and $\tilde{B}u(0) = 0$.

Using (9.38) and (9.39) we obtain the estimate

$$|\tilde{B}u(0)|^2\Delta x \leq K(|g|^2\Delta x + |v_\infty^{(2)}(0)|^2\Delta x + \|F\|^2/r^2) .$$

The value of $v_\infty^{(2)}(0)$ is given by

$$v_\infty^{(2)}(0) = \sum_{\nu=0}^{m-1}(M_\infty^{(2)}(\zeta'))^\nu(T^{-1}(\zeta'))_\infty^{(2)}F(x_\nu) . \tag{9.40}$$

Since the norm of $(T^{-1}(\zeta'))_\infty^{(2)}$ is bounded by $K/r$, the norm $|v_\infty^{(2)}(0)|^2\Delta x$ is bounded by $K\|F\|^2/r^2$, and we arrive at an estimate

$$|Bu(0)|^2\Delta x \leq K(|g|^2\Delta x + \|F\|^2/r^2) .$$

Using the last estimate and (9.38) multiplied by r, we obtain

$$r^2\|u\|^2 + |\tilde{B}u(0)|^2\Delta x \leq K(|g|^2\Delta x + \|F\|^2/r^2) . \tag{9.41}$$

Since $r^2 \geq |z-1| \geq |z|-1$, the last estimate is even stronger than (9.[illegible]) for $|z_j| = 1$. Thus we have proved the sufficiency part of theorem 5.3.

Now assume that only (UKC) is fulfilled. Then (9.36) and (9.39) imply for [illegible] sufficiently small that

$$r\|u\|^2 + |u(0)|^2\Delta x \leq K(|v_\infty^{(1)}(0)|^2\Delta x + |g|^2\Delta x + \|F\|^2/r^3) . \tag{9.42}$$

The value of $v_\infty^{(1)}(0)$ is given by

$$v_\infty^{(1)}(0) = \sum_{\nu=0}^{m-1} (M_\infty^{(1)}(\zeta'))^\nu (T^{-1}(\zeta'))_\infty^{(1)} F(x_\nu) \tag{9.43}$$

and satisfies an inequality

$$|v_\infty^{(1)}(0)|^2 \leq K \sum_{\nu=0}^{m-1} |F(x_\nu)|^2/r^4 = K|F_b|^2/r^4 . \tag{9.44}$$

Since $r^2 \geq |z|-1$, we get from (9.42) and (9.44)

$$(|z|-1)\|u\|^2 \leq K\left(\frac{\|F\|^2}{|z|-1} + \frac{|F_b|^2\Delta x}{r(|z|-1)} + |g|^2\Delta x\right) . \tag{9.45}$$

Then for $|z| > |z_0| = 1 + \alpha_0\Delta x$ with $\alpha_0 > 0$ we have

$$r > |z|-1 > |z_0| - 1 = \alpha_0\Delta x \quad \text{and hence} \quad \Delta x/r < 1/\alpha_0 .$$

Therefore estimate (9.45) is stronger than (6.8) with $|z_0|$ as above.

Let us now prove the necessity part of theorem 5.3. Suppose that (UKC) is not satisfied in $\Omega(\zeta_0')$ and therefore there exists a non-zero vector $(v_0(0), v_I(0))'$ such that

$$\tilde{S}(\zeta_0)(X_0(\zeta_0')v_0(0) + X_I(\zeta_0')v_I(0)) = 0 .$$

Defining with the aid of $v_0(0)$, $v_I(0)$ a homogeneous solution of the equations (9.45) (A), (B), we get from (6.9) with $B = \tilde{B}$ an estimate

$$|\tilde{B}(X_0(\zeta')v_0(0) + X_I(\zeta')v_I(0))|^2 \leq K|g|^2$$

where $g = g(\zeta') = \tilde{S}(\zeta)(X_0(\zeta')v_0(0) + X_I(\zeta')v_I(0))$, so that $g(\zeta_0') = 0$.

Since $\tilde{B}\,X_0(\zeta_0') = 0$ and, as it follows from assumption 9.1, the *columns of* $X_I(\zeta_0')$ are independent of Ker $\tilde{B}$, we conclude that $v_I(0) = 0$ and hence $\tilde{S}(\zeta_0)X_0(\zeta_0')v_0(0) = 0$. It was assumed, however, that $\tilde{S}(\zeta_0)X_0(\zeta_0') \neq 0$. Therefore $v_0(0) = 0$ and (UKC) is proved.

The proof *of condition* 5.1 repeats almost exactly the one used in subsection 8.3 for the case $z_0' \neq 0$. We indicate only the differences. The vector function $\hat{v}(0,\zeta')$ is defined as $r^2v(0)$ instead of $rv(0)$ and $\hat{T}^{-1}(\zeta') = r^2T^{-1}(\zeta')$. Since $(rT^{-1}(\zeta'))_{FI}$ and $(rT^{-1}(\zeta'))^{(2)}_{\infty}$ are analytic, we obtain as before that $\hat{v}_{II}(0,\zeta'), \hat{v}^{(2)}_{\infty}(0,\zeta') = O(r)$. Suppose that $\hat{v}_I(0,\zeta_0')$ in (8.63) is non-zero. Then $\tilde{B}u(0) = \tilde{B}X(\zeta')\hat{v}(0,\zeta')/r^2$, and instead of the estimate $|\tilde{A}u(0)| \ge \delta/r$ we get

$$|\tilde{B}u(0)| = |\tilde{B}X_I(\zeta')\hat{v}_I(0,\zeta')/r^2 + O(1/r)| \ge \delta/r^2 .$$

Then the estimate

$$|\tilde{B}u(0)|^2 \le K r^2/(\Delta x(|z|-|z_0|))$$

implies that

$$r^4/(|z|-|z_0|) \ge \delta > 0$$

for any $|z| > |z_0| = 1+\alpha_0\Delta x$ and any $\Delta x > 0$. Let us define $\zeta = \pi - r\zeta_0'$ and $z = 1+r^2z_0'$. *If r and* $\Delta x$ tend to 0 in such a way that $r^2\,\mathrm{Re}\,z_0' \ge 2\alpha_0\Delta x$, we obtain that $|z| - |z_0| = O(r^2)$ and $r^4/(|z|-|z_0|) = O(r^2) \to 0$. Hence $\hat{v}_I(0,\zeta_0') = 0$ and, as in subsection 8.3, it follows that dim $\tilde{S}(\zeta_0)$Ker $\tilde{B} = 1$. The case Re $z_0' = 0$ is considered exactly as in subsection 8.3.

Let us now consider the case $z_0' = 0$. The operator $F$ in *estimate* (6.9) should be defined as $F(\zeta') = \tilde{F}(\xi)$, where $\xi = \pi - \xi'r$. Theorems 5.1, 5.2 and sufficiency part of theorem 5.3 are formulated locally in $\Omega(\zeta_0')$ in a natural way. The necessity part of theorem 5.3 is formulated as follows:

If estimate (6.9) holds in $\Omega(\zeta_0')$ with $|z_0| = 1+\alpha_0\Delta x$, where $\alpha_0 \ge 0$, and the columns of the matrix $\tilde{S}(\pi,1)(X_0(\zeta_0'), X_I^{(1)}(\zeta_0'))$ are independent, then (UKC) is satisfied in $\Omega(\zeta_0')$, dim $\tilde{S}(\pi,1)$ for $\tilde{P} = 1$ and condition 5.2 holds for any real $\xi = \pi-\xi'r$ for which $\zeta' = (\xi', r, 0) \in \Omega(\zeta_0')$.

Defining the symmetrizer $R(\zeta')$ as in the case $z_0' \neq 0$ we arrive as before at the estimate (9.35). Now the norms of the matrices $r(T^{-1}(\zeta'))_0$, $r(T^{-1}(\zeta'))_\infty^{(1)}$ $(T^{-1}(\zeta'))_{F1}$ and $(T^{-1}(\zeta'))_\infty^{(2)}$ are bounded by $K/|rz'|$ and

$$\|R(\zeta')G\|^2 \le K\|F\|^2/|rz'|^2.$$

As in the case $z_0' \neq 0$ one can show that (UKC) in $\Omega(\zeta_0')$ is equivalent to the condition det $\tilde{S}(\zeta_0)(X_0(\zeta_0'), X_I(\zeta_0')) \neq 0$. Therefore if (UKC) is fulfilled, we get as before the estimate (9.36). If additionally condition 5.1 is satisfied we get the estimate (9.37). Then also estimate (9.38) holds with expression $\|F\|^2/r^3$ replaced by $\|F\|^2/(|z'|^2 r^3)$. Therefore

$$\|u\|^2 \le K\left(\frac{\|F\|^2}{|z'|^2 r^4} + \frac{|g|^2\Delta x}{r}\right)$$

and since $|z|-1 \le |z-1| = |z'r^2|$ and $r \ge |z|-1$ we obtain finally

$$\|u\|^2 \le K\left(\frac{\|F\|^2}{(|z|-1)^2} + \frac{|g|^2\Delta x}{|z|-1}\right) \tag{9.46}$$

i.e. estimate (6.7) with $|z_0| = 1$. Thus theorem 5.2 is proved.

Now assume that only (UKC) is fulfilled. Then in estimate (9.45) one should replace $\|F\|^2/r^3$ by $\|F\|^2/(|z'|^2 r^3)$ and in (9.44) $|F_1|^2/r^4$ by $|F_1|^2/(|z'|^2 r^4)$. Then estimate (9.45) is still valid and estimate (6.8) with $|z_0| = 1+\alpha_0\Delta x > 1$ follows as in the case $z_0' \neq 0$.

Suppose now that in $\Omega(\zeta_0')$ (UKC) and conditions 5.1 and 5.2 are satisfied. We may consider the vector [illegible] in [illegible] as a linear function of g, $v_{II}(0)$ and $v_\infty(0)$ with coefficients analytic in $\Omega(\zeta_0')$. Then the following estimate holds

$$(9.47)\qquad |rz'v_0(0)|^2 + |z'v_I^{(1)}(0)|^2 + |v_I^{(2)}(0)|^2$$

$$\leq K(|rz'v_\infty^{(1)}(0)|^2 + |z'v_{II}^{(1)}(0)|^2 + |v_{II}^{(2)}(0)|^2 + |v_\infty^{(2)}(0)|^2 + |g|^2).$$

In order to prove (9.47) it is enough to show that

$$(9.48)\qquad v_I(0) = 0 \quad \text{if} \quad r = v_{II}(0) = v_\infty^{(2)}(0) = g = 0$$

and

$$(9.49)\qquad v_I^{(2)}(0) = 0 \quad \text{if} \quad z' = v_{II}^{(2)}(0) = v_\infty^{(2)}(0) = g = 0 .$$

Then indeed

$$v_I^{(1)}(0) = O(g, v_{II}(0), v_\infty^{(2)}(0), rv_\infty^{(1)}(0))$$

and

$$v_I^{(2)}(0) = O(g, v_{II}^{(2)}(0), v_\infty^{(2)}(0), z'v_{II}^{(1)}(0), rz'v_\infty^{(1)}(0))$$

and estimate (9.47) follows. The result in (9.48) follows from estimate (9.3[illegible]). Suppose now that the conditions in (9.49) are fulfilled. Then

$$0 = \tilde{S}(\zeta)u(0) = \tilde{S}(\zeta)[X_0(\zeta')v_0(0) + X_{F1}^{(1)}(\zeta')v_{F1}^{(1)}(0) + X_\infty^{(1)}(\zeta')v_\infty^{(1)}(0)]$$

$$+ \tilde{S}(\zeta)X_I^{(2)}(\zeta')v_I^{(2)}(0).$$

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According to parts b) of lemmas 9.1 and 9.3, the columns of $(X_0(\zeta'), X_{F1}^{(1)}(\zeta'), X^{(1)}(\zeta'))$ belong to the space Ker $\tilde{P}(\xi)$. Condition (5.2) and (UKC) imply that the $(n+1)/2$ columns of the matrix $\tilde{S}(\zeta)(X_0(\zeta'),\ X_I^{(1)}(\zeta'))$ form a basis of the space $\tilde{S}(\zeta)(\text{Ker } \tilde{P}(\xi))$. Therefore the vector $\tilde{S}(\zeta)u(0)$ may be represented as a linear combination

$$\tilde{S}(\zeta)u(0) = \tilde{S}(\zeta)X_0(\zeta')w_0 + \tilde{S}(\zeta)X_I^{(1)}(\zeta')w_I^{(1)} + \tilde{S}(\zeta)X_I^{(2)}(\zeta')v_I^{(2)}(0) = 0 \ ,$$

where $w_0$ is a scalar and $w_I^{(1)}$ is a $(n-1)/2$ dimensional column vector. Then (UKC) implies that $w_0 = w_I^{(1)} = v_I^{(2)}(0) = 0$ and estimate (9.47) is proved.

Also the vector $\tilde{P}(\xi)u(0)$ may be considered as a linear function of $g$, $v_{II}(0)$ and $v_\infty(0)$. We shall show that

$$(9.50)\quad |\tilde{P}(\xi)u(0)|^2 \leq K(|g|^2+|v_\infty^{(2)}(0)|^2+|rv_{II}^{(2)}(0)|^2+|rz'v_{II}^{(1)}(0)|^2+|rz'v_\infty^{(1)}(0)|^2) \ .$$

Indeed, if $g = v_\infty^{(2)}(0) = 0$ and $r = 0$, we get as in the case $z_0' \neq 0$ that $\tilde{B}u(0) = 0$. But for $r = 0$, Ker $\tilde{P}(\xi)$ = Ker $\tilde{P}(\pi) \supset$ Ker $\tilde{B}$ and hence $\tilde{P}(\xi)u(0) = 0$. Suppose now that the conditions in (9.49) are fulfilled. Then, according to (9.47), $v_I^{(2)}(0) = 0$ and

$$u(0) = X_0(\zeta')v_0(0) + X_{F1}^{(1)}(\zeta')v_{F1}^{(1)}(0) + X_\infty^{(1)}(\zeta')v_\infty^{(1)}(0) \in \text{Ker } \tilde{P}(\xi).$$

Let us return to equations (7.45) (A), (B) and introduce grid functions $\hat{v}(x)$ and $\hat{G}(x)$ whose components are partitioned according to $v(x)$ and $G(x)$ and given by:

$$\hat{v}_0 = rz'v_0,\ \hat{v}_\infty^{(1)} = rz'v_\infty^{(1)},\ \hat{v}_{F1}^{(1)} = z'v_{F1}^{(1)},\ \hat{v}_{F1}^{(2)} = v_{F1}^{(2)},\ \hat{v}_\infty^{(2)} = v_\infty^{(2)}$$

and $\hat{G}(x)$ is expressed in terms of $G(x)$ in the same way. The matrices $M_j'(\zeta')$ in (9.31) should be replaced by

$$\hat{M}_j' = \begin{vmatrix} M_{j11}' & z'M_{j12}' \\ M_{j21}'/z' & M_{j22}' \end{vmatrix} .$$

According to part d) of lemma 9.3, the matrix $M_{j21}'/z'$ is analytic in $\Omega(\zeta_0')$. Let us denote $\hat{M}_j = -I+r\hat{M}_j'$, $\hat{M}_F = \text{diag}(M_0,\hat{M}_1,\ldots,\hat{M}_t)$. Then equations (7.45) (A), (B) become

$$\text{(9.51)} \quad \begin{aligned} &\text{(A)} \quad (E_x-\hat{M}_F(\zeta'))\hat{v}_F(x) = \hat{G}_F(x) \\ &\text{(B)} \quad (I-M_\infty(\zeta')E_x)\hat{v}_\infty(x) = \hat{G}_\infty(x) . \end{aligned}$$

Let us note that the matrices $\hat{M}_j'(\zeta')$ have the *same eigenvalues* as $M_j'(\zeta')$. Therefore there are symmetrizers $R_j(\zeta')$ such that

$$\text{Re } R_j(\zeta')\hat{M}_j'(\zeta') \leq -\delta I$$

and

$$\hat{v}_j^* R_j(\zeta')\hat{v}_j \geq |\hat{v}_j|^2 \text{ if } \text{Re } \kappa_j' < 0 \text{ and } \hat{v}_j^* R_j(\zeta')\hat{v}_j \geq -c|\hat{v}_j|^2 \text{ if Re } \kappa_j' > 0.$$

Then for sufficiently small $r > 0$ we obtain

$$\hat{M}_j^*(\zeta')R_j(\zeta')\hat{M}_j(\zeta') - R_j(\zeta') \geq \delta r I .$$

Defining

$$R_0(\zeta') = -cI,\ R_\infty(\zeta') = I,\ R_F(\zeta') = \text{diag}(R_0(\zeta'),R_1(\zeta'),\ldots,R_t(\zeta'))$$

we obtain for $\zeta'\in\Omega_R(\zeta_0')$ the estimates (9.33) with $M_F$ replaced by $\hat{M}_F$ and instead of (9.34) we have

$$\hat{v}_F^* R_F(\zeta')\hat{v}_F \geq -c(|\hat{v}_I|^2+|\hat{v}_0|^2) + |\hat{v}_{II}|^2, \quad \hat{v}_\infty^* R_\infty(\zeta')\hat{v}_\infty \geq |\hat{v}_\infty|^2.$$

Applying to equations (9.51) (A), (B) the generalized energy method with the symmetrizers $R_F(\zeta')$ and $R_\infty(\zeta')$ we arrive at estimate (8.67). It follows from definition of $\hat{G}(x)$ and the estimates concerning the rows of $T^{-1}(\zeta')$ that $\|\hat{G}\|^2 \leq K\|F\|^2/r^2$. Estimate (9.47) may be written in a form

$$(9.52) \qquad |\hat{v}_0(0)|^2 + |\hat{v}_I(0)|^2 \leq K(|\hat{v}_{II}(0)|^2 + |\hat{v}_\infty(0)|^2) .$$

Then choosing the constant c in (8.67) small enough and substituting (9.52) in (8.67) we conclude that

$$|\hat{v}_{II}(0)|^2\Delta x \leq K\left(\frac{\|F\|^2}{r^3} + |g|^2\Delta x\right) .$$

Then estimate (9.50) implies that

$$|\tilde{P}(\xi)u(0)|^2\Delta x \leq K\left[\frac{\|F\|^2}{r} + |g|^2\Delta x + (|v_\infty^{(2)}(0)|^2 + |rz'v_\infty^{(1)}(0)|^2)\Delta x\right] .$$

From (9.40) and (9.43) one derives that $|v_\infty^{(2)}(0)|^2\Delta x$ and $|rz'v_\infty^{(1)}(0)|^2\Delta x$ are bounded by $K\|F\|^2/r^2$. Therefore

$$|\tilde{P}(\xi)u(0)|^2\Delta x \leq K\left(\frac{\|F\|^2}{r^2} + |g|^2\Delta x\right) \leq K\left(\frac{\|F\|^2}{|z|-1} + |g|^2\Delta x\right) .$$

Using the last estimate together with (9.46) we obtain finally

$$(|z|-1)\|u\|^2 + |\tilde{P}(\xi)u(0)|^2\Delta x \leq K\left(\frac{\|F\|^2}{|z|-1} + |g|^2\Delta x\right) .$$

Thus we have proved estimate (6.9) with $|z_0| = 1$.

It remains only to prove the necessity part of theorem 5.3. First let us show that (UKC) is satisfied. We proceed as in subsection 8.3 in the case $z_0' = 0$. Supposing that there exists a non-zero vector $(v_0(0), v_I(0))'$ such that

$$\tilde{S}(\zeta)(X_0(\zeta')v_0(0) + X_I(\zeta')v_I(0)) = g(\zeta') \text{ and } g(\zeta_0') = 0$$

we arrive at estimate (8.68) which implies that

$$|\tilde{P}(\xi)X(\zeta')v(0)| \leq K|g(\zeta')|^2 ,$$

where $v_{II}(0) = v_\infty(0) = 0$. Since $\tilde{P}(\pi)X^{(1)}(\zeta_0') = 0$ and, according to assumption 9.3 the columns of $X_I^{(2)}(\zeta_0')$ are independent of $\mathrm{Ker}\tilde{P}(\pi)$, it follows that $v_I^{(2)}(0) = 0$. Therefore $\tilde{S}(\zeta_0)(X_0(\zeta_0')v_0(0) + X_I^{(1)}(\zeta_0')v_I^{(1)}(0)) = 0$. However, we have assumed that the columns of $\tilde{S}(\zeta_0)(X_0(\zeta_0'),X_I^{(1)}(\zeta_0'))$ are independent. Hence $v_0(0) = v_I^{(1)}(0) = 0$ and (UKC) follows. Conditions 5.1 and 5.2 are proved in the same way as in subsection 8.3. The only difference is that estimate (8.69) holds now for all the components of $v_I$.

## 10. Discussion.

In Part II we have investigated a specific difference approximation applied to a very restricted class of mixed initial-boundary value problems with characteristic boundary, while in Part I for the differential case a much wider class of problems was resolved. The question arises: how may this investigation be generalized?

First let us describe the main obstacles which one encounters in the analysis of a multidimensional difference approximation in the non-characteristic case. We suppose that the κ-matrix $L(\kappa,\xi,z)$ associated with the difference scheme is regular for any complex z, $|z| \geq 1$,and real $0 \leq \xi \leq 2\pi$ and has no infinite eigenvalues. Since the work of Gustafsson, Kreiss at al [3] appeared, there seems to be a general acceptance of the idea that the stability theory for the multidimensional case possesses no difficulties, which are not encountered in the one-dimensional case. Let us analyse carefully the theory in [3]. There are two main problems resolved: the first is the block or normal form of thematrix $M(z)$ proved in their Theorems 9.1 and 9.3, and the second consists of the construction of a symmetrizer in Lemma 13.1 for a perturbed Jordan cell in strictly non-dissipative case. The matrix $M(z)$ is obtained from $L(\kappa,z)$ by linearization $\tilde{L}(\kappa,z) = \tilde{A}_0(z) + \kappa\tilde{A}_1(z)$ and then $M(z) = (\tilde{A}_1(z))^{-1}\tilde{A}_0(z)$. Suppose that $|z_0| = 1$ and there are eigenvalues of $L(\kappa,z_0)$ with $|\kappa| = 1$. Theorem 9.1 claims that under Assumptions 5.2 and 5.3 there exists an analytic transformation $T(z)$ such that $T(z)M(z)T^{-1}(z)$ has the block form $\mathrm{diag}(M_1,M_2,\ldots,M_\ell)$ in (9.2) with the matrices $M_j(z)$ as in (9.3)-(9.5). If we recall, for example, the matrix $L(\kappa,\xi,z)$ corresponding to the Burstein difference scheme, then for $\xi = \pi$ this matrix is diagonalizable and thus satisfies Assumption 5.3. However, when $\xi$ is perturbed, the matrix $L(\kappa,\xi,z)$ ceases to be diagonalizable and therefore the block form in Theorem 9.1 may not be maintained. Next, Theorem 9.3 claims that if $M_j(z_0)=\kappa_j I$, where $|z_0| = |\kappa_j| = 1$, and $\|(M_j(z)-\kappa I)^{-1}\| \leq K|z|/(|z|-1)$ for any $|z| > 1$ and $|\kappa| = 1$, then there is a transformation $T_j(z)$ analytic in a neighbourhood of

$z = z_0$ such that

$$T_j^{-1}(z)M_j(z)T_j(z) = \mathrm{diag}(L_j(z),N_j(z))$$

with

$$|z|(L_j^*(z)L_j(z)-I) \leq -\delta(|z|-1)I \ , \ |z|(N_j^*(z)N_j(z)-I) \geq \delta(|z|-1)I.$$

This theorem is entirely "one parametric", i.e. if $M_j$ depends on more parameters, say z and $\xi$, then the theorem does not hold any more. Actually, in order to get an appropriate block form for the matrix $M(\xi,z)$ near the point $(\xi_0,z_0)$, one should provide an additional parametrization of $z-z_0$ and $\xi-\xi_0$ as we have done in Sections 8 and 9. However the success of such parametrization can not be guaranteed.

Now let us analyse the construction of the symmetrizer in Lemma 13.1. Because of the strict non-dissipativity, the double-sided resolvent condition (13.6) holds and the existence of the symmetrizer follows easily by Ralston's note. However, in multidimensional case such a symmetrizer should be constructed also for dissipative schemes when the resolvent condition (13.6) is not valid. Our theorem 8.1 in subsection 8.2 actually solves this problem.

Suppose that $L(\kappa,\xi,z)$ corresponds to a dissipative difference scheme (Burstein scheme is not completely dissipative, e.g. at $\xi = \pi$). Then the eigenvalues $|\kappa_j| = 1$ are possible only when $z = 1$, $\xi = 0$ and $\kappa_j = 1$. The investigation of the block structure performed by us in subsection 8.1 may be applied to a general dissipative difference scheme. Thus, together with our theorem 8.1 it provides a complete solution (in the terms of the uniform Kreiss condition) for the stability of a dissipative difference approximation applied to strictly hyperbolic problems with non-characteristic boundary.

If the boundary is characteristic, in addition to the difficulties described above, one faces the perturbation problem for a singular $\kappa$-matrix. There is no general theory for this case. However, the following uniform singularity condition for the $\kappa$-matrix $L(\kappa,\xi,z)$ seems to be essential: if the determinant $|L(\kappa,\xi,z)| \equiv 0$ for some $\xi_0$ and $z_0$, then there is some analytic line $z = f(\xi)$, $0 \leq \xi \leq 2\pi$, with $z_0 = f(\xi_0)$ such that the determinant vanishes identically along this line. For example, the Friedrichs type schemes or the original Lax-Wendroff scheme do not satisfy this condition even in the case $|A\alpha+B\beta| \equiv 0$, and because of that the corresponding $\kappa$-matrices do not have an analytic block structure. The same problem arises with the Burstein difference approximation in the case $|-\beta bI+A\alpha+B\beta| \equiv 0$ where $b = \text{const} \neq 0$ - this is, for example, the case of the shallow water equations with matrices

$$A = \begin{pmatrix} 0 & c & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} b & 0 & 0 \\ 0 & b & c \\ 0 & c & b \end{pmatrix}.$$

However, the leap-frog scheme in this case possesses no difficulties. Although it is hard to develop a general stability theory in the characteristic case, the methods used in this work may be applied to any difference scheme with corresponding matrix $L(\kappa,\xi,z)$ being a polynomial of some linear combination $\alpha A+\beta B$ and satisfying the uniform singularity condition.

References

[1] A.Majda and S. Osher, Initial-boundary value problems for hyperbolic equations with uniformly characteristic boundary,
Comm. Pure Appl. Math., Vol. 28, 1975, pp. 607-675.

[2] H.-O. Kreiss, Initial boundary value problems for hyperbolic systems,
Comm. Pure Appl. Math., Vol. 23, 1970, pp. 277-298.

[3] B. Gustafsson, H.-O. Kreiss and A. Sundström, Stability theory of difference approximation for mixed initial boundary value problems, II.,
Math. Comp., Vol. 26, 1972, pp. 649-686.

[4] I. Gohberg and L. Rodman, On spectral analysis of non-monic matrix and operator polynomials, I.,
Israel Jour. Math. Vol. 30, Nos. 1-2, 1978, pp. 133-151.

[5] I. Gohberg, M.A. Kaashoek, and D.C. Lay, Equivalence, linearization and decomposition of holomorphic operator functions,
J. Func. Anal., Vol. 28, 1978, pp. 102-144.

[6] V.I. Arnold, On matrices depending on parameters,
Russ. Math. Surveys, Vol. 26, No. 2 (1971), pp. 29-43.

[7] M.S. Agranovic, Theorem of matrices depending on parameters and its applications to hyperbolic systems.
Func. Anal. & Appl., Vol. 6, No. 2 (1972), pp. 85-93.

[8] J.V. Ralston, Note on a paper of Kreiss,
Comm. Pure Appl. Math., Vol. 24, 1971, pp. 759-761.

[9] R.C. Gunning, H. Rossi, Analytic functions of several complex variables,
Englewood Cliffs, N.J., Prentice-Hall, (1965).

[10] F.R. Gantmacher, The theory of matrices,
Vol. II, chap. XII, Chelsea Publishing Company, N.Y. (1960).

## List of Corrections

| place | written | should be |
|---|---|---|
| p. 4, ℓ. 13 from the top | notion | notation |
| p. 7, ℓ. 5 " " " | $\omega/\|\lambda\|$ | $\omega/\|\zeta\|$ |
| p. 22, ℓ. 10 " " " | $,\varphi^{(n)}(\lambda))$ | $,\varphi^{(n)}(\lambda))'$ |
| p. 29, ℓ. 4 " " " | $\lambda).$ | $\lambda').$ |
| p. 29, ℓ. 2 from the bottom | $\lambda$-roots | $\lambda'$-roots |
| p. 34, ℓ. 3 " " " | $\lambda_j, q_j$ | $\lambda'_j, q_j$ |
| p. 35, formula (3.17) | $v^{(n-1)'}$ | $v^{(n-1)})'$ |
| p. 36, ℓ. 1 from the top | $T'$ | $T^{-1}$ |
| p. 41, at the bottom | $N'_0$ | $N_0$ |
| p. 44, ℓ. 5 from the bottom | $X(\zeta')$ | $X_j(\zeta')$ |
| p. 46, ℓ. 2 from the top | $(1,0,\ldots,0)$ | $(1,0,\ldots,0)'$ |
| p. 52, ℓ. 4 from the bottom | all | whole |
| p. 60, ℓ. 6 from the top | is | in |
| p. 61 formula (4.6) (B) | $G_\infty^{(2)}$ | $-G_\infty^{(2)}$ |
| " (C) | $-G_\infty^{(1)}$ | $+G_\infty^{(1)}$ |
| p. 63 formula (4.11) (B) | $y_\infty^{(1)}$ | $y_\infty^{(2)}$ |
| p. 68, ℓ. 10 from the top | $\\|f_b\\|_{x,y}^2$ | $\\|f_b\\|_y^2$ |
| p. 72, ℓ. 13, 15 " " " | (5.21) | (5.22) |
| p. 74, ℓ. 13 " " " | (5.22) | (5.23) |
| p. 82, ℓ. 15 " " " | $\lambda_1,\lambda_2,\ldots,\lambda_{n-1}$ | $\lambda_2,\lambda_3,\ldots,\lambda_n$ |
| p. 82, ℓ. 17, 18 " " " | $j = 1, \quad n-1$ | $j = 2, \quad n$ |
| p. 84, ℓ. [illegible] " " " | $\|\kappa_j\| \neq 0$ | $\|\kappa_j\| \neq 1$ |
| p. 84, ℓ. [illegible] " " " | have $\|\kappa_j\| > 1$ | have $\|\kappa_j\| < 1$ |
| p. 99 formula (7.31) | $\tilde{X}^{(1,2)}$ | $\tilde{X}_\infty^{(1,2)}$ |

# List of Corrections

| place | written | should be |
|---|---|---|
| p. 100, ℓ. 3 from the bottom | $\kappa^{m+3-k}$ | $\kappa^{m-3-k}$ |
| p. 100, ℓ. 1 " " " | $n\times(\rho-1)$ | $n\times[(\rho-1)\cdot(m-2)]$ |
| p. 102 formula (7.35) | $(0,\tilde{M}_\infty^{(2,1)}(\zeta))'$ | $(\tilde{M}_\infty^{(2,1)}(\zeta),0)'$ |
| p. 121, ℓ. 9 from the bottom | $1+r\kappa_j$ | $1+r\kappa_j'$ |
| p. 126, ℓ. 2 from the top | (8.22) | empty |
| p. 126, ℓ. 12 " " " | $B_0$ and $B_1$ | $\tilde{B}_0$ and $\tilde{B}_1$ |
| p. 126, ℓ. 14 " " " | (8.23) | (8.22) |
| p. 126, ℓ. 19 " " " | $\Omega(\zeta_0)$ | $\Omega(\zeta_0')$ |
| p. 127, ℓ. 6 from the bottom | (8.23) | (8.22) |
| p. 134, ℓ. 6, 7 from the top | $\mathrm{Im}\,\kappa'$ | $\mathrm{Im}\,\hat{\kappa}'$ |
| p. 137, ℓ. 1,4,5,7 from the bottom | $\kappa$ | $\hat{\kappa}'$ |
| p. 139, formula (8.44) | $\partial\kappa'$ | $\partial\hat{\kappa}'$ |
| p. 143, ℓ. 4 from the bottom | $q(z_1)$ | $g(z_1)$ |
| p. 144, ℓ. 17, 18 from the top | in a natural way when Re $\kappa_j' = 0$. | in a natural way into groups I and II, where definitions (8.28) and (8.29) are used in the case when Re $\kappa_j' = 0$. |
| p. 150, ℓ. 10 from the top | $u_0$ | $u(0)$ |
| p. 152, formula (8.67) | $]$ | $)]$ |
| p. 154, ℓ. 2 from the top | vector $w$ | vector $\varphi$ |
| p. 155, ℓ. 4 " " " | 5.2 | [illegible] |
| p. 161, ℓ. 8 | corresponding to | correspond |
| p. 161, formula (9.15) | $P$ | $P_1$ |
| p. 163, formula (9.22) | $Q$ | $Q_j$ |
| p. 167, ℓ. 2 from the bottom | $M_\infty^{(2)}$ | $\kappa M_\infty^{(2)}$ |
| p. 175, ℓ. 14 from the top | $\zeta = (\zeta_1, z)$ | $\zeta = (\xi_1, z)$ |
| p. 181, ℓ. 6 from the bottom | $v(0)$ | $v_\infty(0)$ |

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